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An Introduction to some properties of the pseudo arc

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AN INTRODUCTION TO SOME PROPERTIES OF THE PSEUDO ARC

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CHAPTER 1 Introduction

In the early 1900's several topologists studied some rather unusual continua, called indecomposable continua. A continuum is a compact, connected metric space. As their name implies, **indecomposable continua** can not be decomposed, that is they can not be written as the union of two proper subcontinua. As is often the case in topology, the definition is easily understood, but the examples can be difficult to find and rather bizarre.

Perhaps the first published example of an indecomposable continuum was Brouwer's [7] in 1910. In 1917 Yoneyama [18] published his famous example, the Lakes of Wada. Knaster's bucket handles is one of the most easily constructed of all the indecomposable continua. To construct it, one simply joins the points of the Cantor set with semicircles as shown in Figure 1-1. The resulting continuum can be shown to be indecomposable.

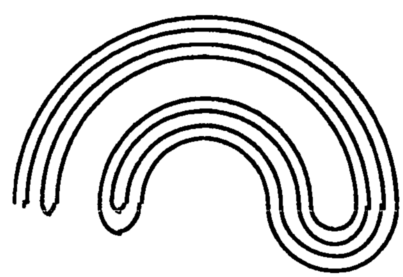


Figure 1-1

One of the most interesting, and least well known, of all the indecomposable continua is the pseudo arc. The pseudo arc is actually **hereditarily indecomposable,** meaning each of its nondegenerate subcontinua is also indecomposable. Note that Knaster's bucket handles is not hereditarily indecomposable since every proper subcontinuum is an arc.

In 1921Mazurkiewicz[12] asked if every plane continuum which is homeomorphic to each of its subcontinua is an arc. A continuum with this property is said to be **hereditarily equivalent**. As an answer, Moise [14] constructed a hereditarily indecomposable continuum, which was similar to that discovered by Knaster[10] years earlier, and proved it to be hereditarily equivalent. Because this continuum shared a property with an arc, but was not an arc, Moise named it a pseudo arc.

When Knaster [10] described his hereditarily indecomposable continuum he suggested that it might be homogeneous. A continuum is **homogeneous** if for any two points of the continuum there exists a homeomorphism that maps one point to the other. An arc is not homogeneous because any homeomorphism from the arc onto itself cannot map the end points to any interior points. A circle, a sphere, and a torus are all examples of a continua that are homogeneous, since any point can be mapped homeomorphically into any other point by rotation. In about 1920 Knaster and Kuratowski [11] posed the following question: If a nondegenerate plane continuum is homogeneous, is it necessarily a simple closed curve? Waraszkiewicz [15] and Choquet [8] claimed that the answer was yes. So it came as quite a surprise when Bing [3], after hearing a talk by Moise, proved the pseudo arc was homogeneous. A year later, in 1950, Bing [4] was able to prove that every hereditarily indecomposable continuum is a pseudo arc if it is chainable. So, in fact, Knaster's hereditarily indecomposable continuum is also a pseudo arc.

Thus, the pseudo arc has some properties of both an arc and a circle; namely, it is hereditarily equivalent and homogeneous. However, it also has some properties that neither the arc nor the circle have. One example, proved by Bellamy and Lysko[2], is that the product of two pseudo arcs is **factorwise rigid**. This means every homeomorphism of a product of two pseudo arcs can be written as a product of two homeomorphisms of the factors.

In this thesis we will concentrate on the homogeneity of the pseudo arc as developed in the papers [3], [4], and [5] of Bing. In chapter 2 we will explain the construction of the pseudo arc and prove it is hereditarily indecomposable. The construction used is the same as Bing's [3] with some added notation to help the reader. Bing's construction, although similar, was an improvement over Moise's [14]. Chapter 3 deals with properties of chains; the families of sets used by Bing and Moise to construct the pseudo arc. Again, new notation and vocabulary are introduced to improve the readability. Some of the properties that Bing included in [3] have been removed from this chapter, since they are not needed in subsequent chapters; however, they have been included in an appendix for the interested reader. The thesis culminates in chapter 4 where we prove that the pseudo arc is homogeneous. It will also be shown that any chainable continuum which is hereditarily indecomposable or homogeneous is a pseudo arc. As a corollary to these final theorems we can state that the pseudo arc is hereditarily equivalent.

New notation will be introduced at various points in the thesis. For easy reference the standard notations that are used are listed below.

i) Script letters represent families of sets.

ii) Capital letters represent sets.

iii) Lower case letters represent points.

iv) $\bigcup \mathcal{D} = \bigcup \{D \mid D \in \mathcal{D}\}$

v) d(a, b) is the Euclidian distance between points a and b.

vi) $N_\varepsilon(A) = \{x \mid d(a, x) < \varepsilon \text{ for every } a \in A\}$ is the ε-neighborhood of A.

vii) $d(A, B) = \inf\{ \varepsilon > 0 \mid A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}$ denotes the Hausdorff distance between sets A and B. Note: If A and B are points then the Hausdorff distance is the same as the Euclidian distance. Also, if $A \cap B = \emptyset$, some authors use d(A, B) to mean the shortest distance between A and B. But this is not the same as the Hausdorff distance between A and B. In this thesis, we will never use d(A, B) to denote the shortest distance between A and B.

viii) $\text{diam}(A) = \sup\{ d(a, b) \mid a, b \in A\}$ is the diameter of a set A.

Some preliminary definitions and theorems (given without proof) from continuum theory may also be helpful.

DEFINITION: The **composant** of a point p in X is the set of all points that can be joined with p by a proper subcontinuum of X.

Example: Let X = [0, 1]. The composant of the point 0 is [0, 1). The composant of the point 1 is (0, 1]. The composant of any point in (0, 1) is [0, 1]. Notice that, in general, the composants do not partition the space. However, we have the following theorem regarding the composants of indecomposable continua.

THEOREM 1.1: An indecomposable continuum has uncountably many disjoint composants.

DEFINITION: Let a and b be points of a continuum X. If no proper subset of X contains both a and b, then X is said to be **irreducible between a and b**. The point a is said to be a **point of irreducibility**.

Example: [0, 1] is irreducible between 0 and 1. Thus, 0 and 1 are both points of irreducibility. Let x be any other point in (0, 1), and let y be any other point in [0, 1].

Since the interval [x, y] is a proper subset of [0, 1], x is not a point of irreducibility.

THEOREM 1.2: A continuum is indecomposable if and only if every point of the continuum is a point of irreducibility.

THEOREM 1.3: Let $\{K_i \mid K_i \neq \emptyset$ for every $i = 1, 2, \ldots\}$ be a collection of continua such that, for every $i = 1, 2, \ldots, K_{i+1} \subset K_i$. Then $\bigcap_i K_i$ is a continuum.

The above theorem is an adaptation of a theorem from page 203 of Willard[17].

CHAPTER 2 Construction of the Pseudo Arc

Bing, refining Moise's work, constructed his pseudo arc using a descending sequence of chains, where each chain is a collection of sets. In order to make the intersection a hereditarily indecomposable continuum, each chain in the sequence is "crooked" with respect to the previous chain. The result is a continuum so crooked that there are no decomposable subcontinua. We will begin the construction with some definitions.

DEFINITION: A **chain** $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ from p to q is a collection of open sets, called **links**, where $p \in D_1$, $q \in D_n$, and $D_i \cap D_j \neq \emptyset$ if and only if i and j are consecutive integers. D_1 and D_n are **endlinks**, and D_i, i = 2, 3, . . , n-1 are **interior links**. A subchain of $\mathcal{D}$ with end links D_h and D_k is denoted $\mathcal{D}(h, k)$. (Although h and k are such that $h < k$ or $h > k$, for convenience we assume, unless otherwise stated, that $h < k$ throughout this thesis.)

Note that there is no requirement that the links themselves be connected, only that adjacent links intersect.

DEFINITION: A chain $\mathcal{E}$ is a **refinement** of a chain $\mathcal{D}$ if each link of $\mathcal{E}$ is contained in a link of $\mathcal{D}$, denoted by $\mathcal{E} < \mathcal{D}$.

In Figure 2-1 the chain $\mathcal{E}$ is a refinement of $\mathcal{D}$.

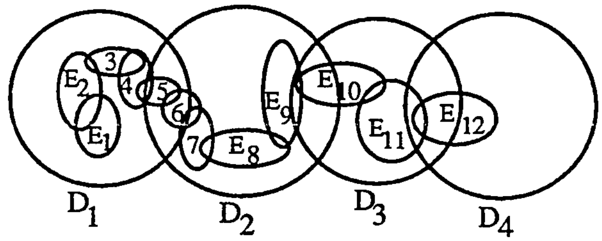


Figure 2-1

The following definition shows how the specific "shape" of one chain can be described by the pattern it follows as a refinement of a larger chain. If $\mathcal{E} < \mathcal{D}$, then the pattern can be considered as a function that assigns, with respect to containment, each link of $\mathcal{E}$ to a link of $\mathcal{D}$.

DEFINITION: A **pattern** is a function $f:\{1, 2, 3, \ldots, n\} \to \mathbb{N}$ such that $|f(i) - f(i+1)| \leq 1$, for every $i = 1$ to n.

DEFINITION: A chain $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ **follows the pattern** $f: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ in a chain $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ if $E_i \subset D_{f(i)}$, for every $i = 1, 2, \ldots, n$.

For example, the chain $\mathcal{E}$ in Figure 1 follows the pattern $f(1) = 1$, $f(2) = 1$, $f(3) = 1$, $f(4) = 1$, $f(5) = 1$, $f(6) = 2$, $f(7) = 2$, $f(8) = 2$, $f(9) = 2$, $f(10) = 3$, $f(11) = 3$, and $f(12) = 4$ in $\mathcal{D}$. More simply, the pattern f can be described as a sequence of range values in the order in which they are used. Thus, we say that $\mathcal{E}$ follows the pattern 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 4 in $\mathcal{D}$.

The obvious question now is: what pattern would a chain $\mathcal{E}$ have to follow in $\mathcal{D}$

in order for $\mathcal{E}$ to be crooked? It is tempting to look at Figure 2-1 and say that $\mathcal{E}$ is crooked; however, Figure 2-2 shows a straight chain $\mathcal{F}$ that follows the same pattern in $\mathcal{D}$. So, $\mathcal{E}$ is not crooked in $\mathcal{D}$ because all of its bends and turns are contained in single links of $\mathcal{D}$ in such a way that they can be straightened out, without changing the pattern that $\mathcal{E}$ follows in $\mathcal{D}$. To be crooked, then, the chain $\mathcal{E}$ will have to pass back and forth through the links of $\mathcal{D}$.

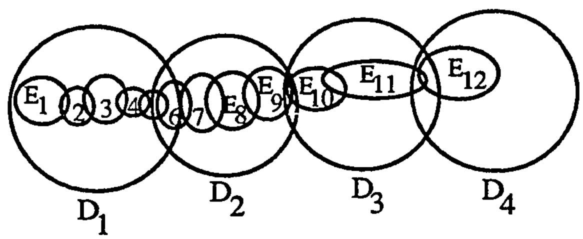


Figure 2-2

DEFINITION: A chain $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ is **crooked** in a chain $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$, denoted $\mathcal{E} \precsim \mathcal{D}$, provided

1. $\mathcal{E} < \mathcal{D}$

2. If a subchain $\mathcal{E}(i, j)$ is such that $E_i \cap D_h \neq \varnothing$, $E_j \cap D_k \neq \varnothing$ and $|h - k| > 2$, then $\mathcal{E}(i, j) = \mathcal{E}(i, r) \cup \mathcal{E}(r, s) \cup \mathcal{E}(s, j)$ such that $i < r < s < j$, $E_r \subset D_{k-1}$, and $E_s \subset D_{h+1}$.

Note that $|h - k| > 2$ means $\mathcal{D}(h, k)$ must have 4 or more links.

Thus, the subchain $\mathcal{E}(i, j)$ follows a pattern that starts in D_h, passes through D_{k-1}, comes back to D_{h+1}, and then ends in D_k. As a result, when $\mathcal{E}$ is crooked in $\mathcal{D}$, each subchain of $\mathcal{E}$ which is contained in a 4 or more linked subchain of $\mathcal{D}$ is crooked in $\mathcal{D}$.

Also, note that $\mathcal{E}(i, r) \lesssim \mathcal{D}(h, k\text{-}1)$, $\mathcal{E}(r,s) \lesssim \mathcal{D}(h+1, k\text{-}1)$, and $\mathcal{E}(s, j) \lesssim \mathcal{D}(h+1, k)$.

Example: To build an example of a crooked chain, we will start with a chain $\mathcal{D}$ of 4 links and a chain $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ such that $\mathcal{E} < \mathcal{D}$, $E_1 \cap D_1 \neq \varnothing$, and $E_n \cap D_4 \neq \varnothing$. For $\mathcal{E}$ to be crooked in $\mathcal{D}$ there must be a link $E_r \subseteq D_3$ and a link $E_s \subseteq D_2$, where $1 < r < s < n$, as shown in Figure 2-3. For figures 2-3, 2-4a, and 2-4b, let us assume that the lines joining the given links give the exact path of the chain. In general this would not be true, but here we are constructing a particular example. Before we can be sure that $\mathcal{E}$ is crooked in $\mathcal{D}$ the other subchains of $\mathcal{E}$ must be shown to meet the requirements of the definition. Notice, however, that by our construction the others subchains, $\mathcal{E}(1, r)$ for example, are contained in fewer than 4 links of $\mathcal{D}$, so the definition is satisfied.

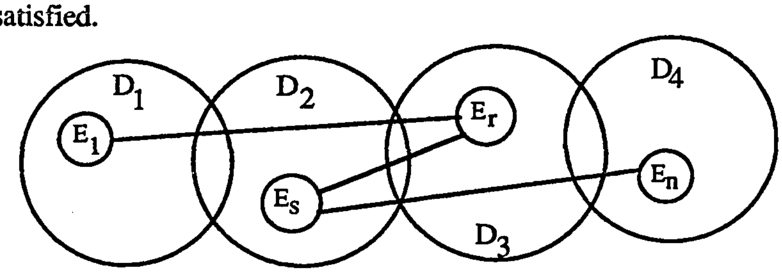


Figure 2-3

As the chain $\mathcal{D}$ grows in length the construction of $\mathcal{E}$ becomes more complicated. If $\mathcal{D}$ had 5 links, then Figure 2-4a shows that $\mathcal{E}(1,r)$ would be contained in a 4 link subchain of $\mathcal{D}$. Therefore, $\mathcal{E}(1,r)$ would have to meet the requirements of the definition. We can make the subchain $\mathcal{E}(1,r)$ follow the same pattern in $\mathcal{D}(1, 4)$ that the chain $\mathcal{E}$ followed when $\mathcal{D}$ had only 4 links. Likewise, as shown in Figure 2-4b, the subchain $\mathcal{E}(s,n)$ will follow a similar pattern in $\mathcal{D}(2, 5)$, and the subchain $\mathcal{E}(r,s)$, which is only contained in 3 links of $\mathcal{D}$, will consist of a straight chain. In fact, it is now easy to see that the construction of a crooked chain is a somewhat recursive process. Given that $\mathcal{D}$ has m links, in order for $\mathcal{E}$ to be crooked in $\mathcal{D}$, $\mathcal{E}$ must have 2 subchains, $\mathcal{E}(1,r)$ and $\mathcal{E}(s,n)$, each crooked in m - 1 links of $\mathcal{D}$ and one subchain, $\mathcal{E}(r,s)$, crooked in m - 2 links of $\mathcal{D}$. Specifically, $\mathcal{E} \lesssim \mathcal{D}$ means $\mathcal{E}(1, r) \lesssim \mathcal{D}(1, m - 1)$, $\mathcal{E}(r, s) \lesssim \mathcal{D}(2, m - 1)$, and $\mathcal{E}(s, n) \lesssim \mathcal{D}(2, m)$.

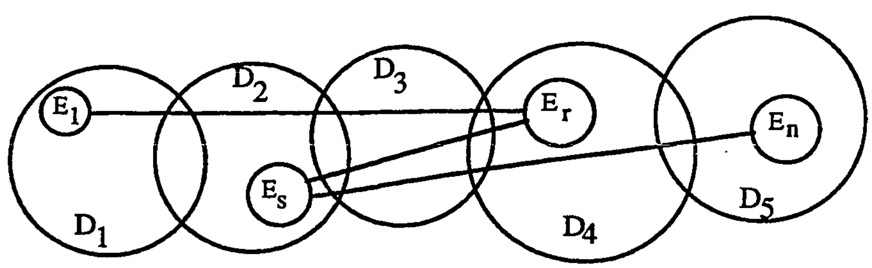


Figure 2-4a

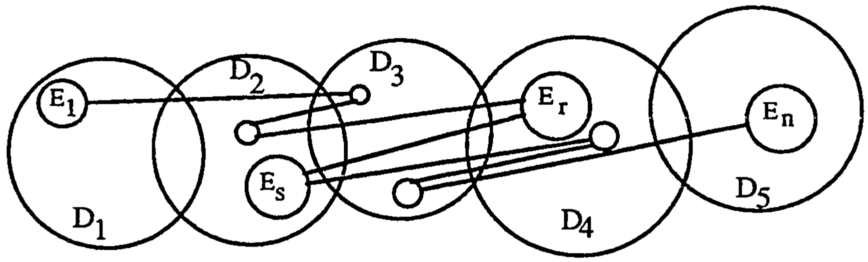


Figure 2-4b

DEFINITION: Let p and q be points in the plane. A sequence of chains $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3, \ldots$ from p to q is said to be a **crooked sequence** if for each positive interger i:

1) $\mathcal{D}_{i+1} \lesssim \mathcal{D}_i$,

2) Every link of $\mathcal{D}_i$ has diameter less than 1/i, and

3) The closure of each link of $\mathcal{D}_{i+1}$ is a subset of a link of $\mathcal{D}_i$.

When using a crooked sequence of chains we will denote the j^{th} link of $\mathcal{D}_i$ by $D(i)_j$. A pseudo arc is constructed using a crooked sequence of chains, as shown in the following definition.

DEFINITION: Let $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3, \ldots$ be a crooked sequence of chains from p to q. The **pseudo arc** is $\bigcap_i (\mathrm{Cl}(\bigcup \mathcal{D}_i))$. Throughout this paper we will use M to denote a pseudo arc. The points p and q are called **endpoints.**

It should be noted that the crooked sequence in the above definition is not unique. Later, we will show that it does not matter what crooked sequence is used, since we will prove that all pseudo arcs are homeomorphic to one another. In particular, let $\mathcal{D}_1$, $\mathcal{D}_2$,. . be such that the links of each chain is connected, then $\mathcal{D}_1$, $\mathcal{D}_2$, . . is a collection of nested continua. If we apply Theorem 1.3 to this particular case we can see that the pseudo arc is a continuum.

THEOREM 2.1: A pseudo arc is hereditarily indecomposable.

Proof: To prove this theorem, we must prove that every subcontinuum of the pseudo arc is indecomposable. Suppose this isn't true and let N be decomposable subcontinuum of M. Then $N = H \cup K$, where H and K are proper subcontinua of N. Also, there exist points $p \in K - H$, $q \in H - K$ and some integer j such that $d(p, H) > 2/j$, and $d(q, K) > 2/j$.

Let $\mathcal{D}_j(u, v)$ be a subchain of $\mathcal{D}_j$ from p to q. Without loss of generality, let $p \in D(j)_u$. Since N is a continuum, each link of $\mathcal{D}_j(u, v)$ contains some points of N. However, the diameter of each link of $\mathcal{D}_j$ is less than $1/j$ and the $d(p, H) > 2/j$, thus, $H \cap D(j)_{u+1} = \emptyset$. Using similar reasoning, $K \cap D(j)_{v-1} = \emptyset$. This implies that $v - u > 2$, or, in other words, $\mathcal{D}_j(u, v)$ has at least 4 links. Let $\mathcal{D}_{j+1}(h, k)$ be a subchain of $\mathcal{D}_{j+1}$ such that $\mathcal{D}_{j+1}(h, k) < \mathcal{D}_j(u, v)$, $D(j+1)_h \subset D(j)_u$, $D(j+1)_k \subset D(j)_v$, and each link of $\mathcal{D}_{j+1}(h, k)$ meets N.

By the construction of M, $\mathcal{D}_{j+1}(h, k) \lesssim \mathcal{D}_j(u, v)$, which means there exists links $D(j+1)_r$, $D(j+1)_s$, and $D(j+1)_t$ such that $h < r < s < t < k$, $D(j+1)_s \subset D(j)_{u+1}$, and $D(j+1)_r \cup D(j+1)_t \subset D(j)_{v-1}$ (see figure 2-5).

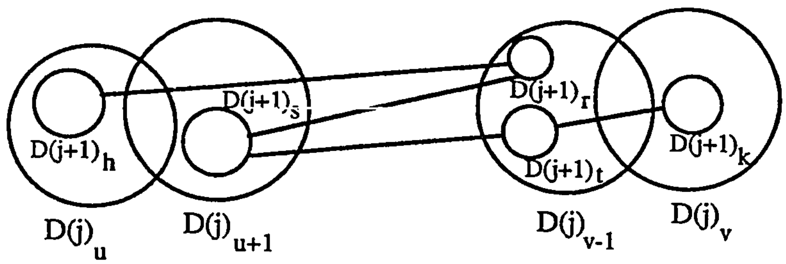


Figure 2-5

Since $K \cap D(j)_{v-1} = \emptyset$, H must contain the points of N in $D(j+1)_r$ and $D(j+1)_t$. $H \cup D(j)_{u+1} = \emptyset$ so H does not contain any points in $D(j+1)_s$. But, N passes from D(j+1)r, through D(j+1)s, to D(j+1)t. Thus, H is not a continuum, which is a contradiction. Hence, N is indecomposable and M is hereditarily indecomposable. ■

CHAPTER 3 Properties of Crooked Chains

In this chapter, we will concern ourselves with proving some properties of crooked chains. Later, in Chapter 4 these properties will be used to prove the homogeneity of the psuedo arc. In many of the theorems that follow the proofs involve making a new chain by combining the links of some other chain.

DEFINITION: A chain $\mathcal{E}$ is a **consolidation** of a chain $\mathcal{D}$ if $\mathcal{D} < \mathcal{E}$ and each link of $\mathcal{E}$ is a union of links of $\mathcal{D}$, denoted by $\mathcal{E} \underset{c}{>} \mathcal{D}$ or $\mathcal{D} \underset{c}{<} \mathcal{E}$. (Some authors refer to consolidations as amalgamations.)

Note that if $\mathcal{D} \underset{c}{<} \mathcal{E}$, then $\mathcal{D} < \mathcal{E}$.

Example: Suppose we are given the chain $\mathcal{D} = \{D_1, D_2, D_3, D_4, D_5, D_6\}$ as in Figure 3-1. If $E_1 = D_1 \cup D_2$, $E_2 = D_3 \cup D_4$, and $E_3 = D_5 \cup D_6$, then $\mathcal{E} = \{E_1, E_2, E_3\} \underset{c}{>} \mathcal{D}$.

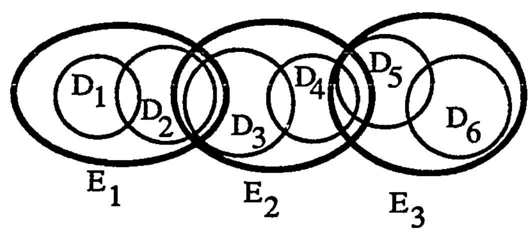


Figure 3-1

It is a mistake to think that when consolidating a chain we must only take the union of adjacent links. Since each link of $\mathcal{E}$ does not have to be connected, we could have said $E_1 = D_3$, $E_2 = D_2 \cup D_4$, and $E_3 = D_1 \cup D_5 \cup D_6$ and still $\mathcal{E} = \{E_1, E_2, E_3\} \underset{c}{>} \mathcal{D}$. This consolidation is shown in Figure 3-2.

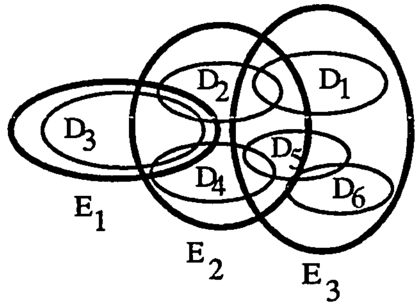


Figure 3-2

It is easy to see that the transitive property holds for the relations $<$, $\lesssim$, and $\underset{c}{\lesssim}$. For example, if $\mathcal{F} \lesssim \mathcal{E}$ and $\mathcal{E} \lesssim \mathcal{D}$, then $\mathcal{F} \lesssim \mathcal{D}$. But, what if we replace $\mathcal{F} \lesssim \mathcal{E}$ by $\mathcal{F} < \mathcal{E}$ or $\mathcal{F} \underset{c}{\lesssim} \mathcal{E}$, or what if we replace $\mathcal{E} \lesssim \mathcal{D}$ by $\mathcal{E} \underset{c}{\lesssim} \mathcal{D}$? Would the above statement still be true? The next two theorems and the corollary answer this question affirmatively. It might be interesting to find other "transitivity" theorems; however, for our purposes these are all that is needed.

THEOREM 3.1 If $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ are chains such that $\mathcal{F} \lesssim \mathcal{E}$ and $\mathcal{E} \underset{c}{\lesssim} \mathcal{D}$, then $\mathcal{F} \lesssim \mathcal{D}$.

Proof: Let $\mathcal{F}(h, k) < \mathcal{D}(r, s)$ be such that $h < k$, $r < s$, $|r - s| > 2$, $F_h \cap D_r \neq \emptyset$, $F_k \cap D_s \neq \emptyset$, and no interior link of $\mathcal{F}(h, k)$ intersects D_r or D_s.

Since $\mathcal{D} \underset{c}{\gtrsim} \mathcal{E}$, each link of $\mathcal{D}$ is the union of links of $\mathcal{E}$. Thus, there exists a subchain $\mathcal{E}(m, n)$ such that $\mathcal{F}(h, k) < \mathcal{E}(m, n)$, $E_m \subset D_r$, $E_n \subset D_s$, $F_h \cap E_m \neq \emptyset$, and $F_k \cap E_n \neq \emptyset$. But $E_m \subset D_r$ and $E_n \subset D_s$, so no interior link of $\mathcal{F}(h, k)$ intersects E_m or E_n.

Because $\mathcal{F} \lesssim \mathcal{E}$, there exists the links F_u and F_v of $\mathcal{F}$ such that $F_u \subset E_{n-1} \subset D_{r+1}$, $F_v \subset E_{m+1} \subset D_{s-1}$, and $h < u < v < k$. Therefore, $\mathcal{F} \lesssim \mathcal{D}$. ■

THEOREM 3.2 If $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ are chains such that $\mathcal{F} < \mathcal{E}$, and $\mathcal{E} \lesssim \mathcal{D}$, then $\mathcal{F} \lesssim \mathcal{D}$.

Proof: Suppose $F_h \subset D_r$, $F_k \subset D_s$ and $|r - s| > 2$. Since $\mathcal{F} < \mathcal{E}$, there are links E_m and E_n such that $F_h \subset E_m$ and $F_k \subset E_n$.

$\mathcal{E} \lesssim \mathcal{D}$, so $\mathcal{E}(m, n) = \mathcal{E}(m, p) \cup \mathcal{E}(p, q) \cup \mathcal{E}(q, n)$, with $m < p < q < n$, $E_p \subset D_{s-1}$, and $E_q \subset D_{r+1}$. The subchain $\mathcal{F}(h, k)$ passes from E_m to E_n, thus there is some link of $\mathcal{F}(h, k)$ that is contained in E_p, call it F_u. Since $p < q$, there is a link of $\mathcal{F}(u, k)$ contained in E_q, call it F_v.

Therefore, $\mathcal{F}(h, k) = \mathcal{F}(h, u) \cup \mathcal{F}(u, v) \cup \mathcal{F}(v, k)$, $h < u < v < k$, $F_u \subset D_{s-1}$, and $F_v \subset D_{r+1}$. Hence, $\mathcal{F} \lesssim \mathcal{D}$. ■

Since $\mathcal{F} \underset{c}{\lesssim} \mathcal{E}$ means that $\mathcal{F} < \mathcal{E}$, we have the following corollary to Theorem 3.2.

COROLLARY : If $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ are chains such that $\mathcal{F} \underset{c}{\lesssim} \mathcal{E}$, and $\mathcal{E} \lesssim \mathcal{D}$, then $\mathcal{F} \lesssim \mathcal{D}$.

In Chapter 4, in order to prove that the psuedo arc is homogeneous, we will show that every point of the psuedo arc is essentially an "endpoint." This does not mean, however, that every link of the chains used to construct the pseudo arc can be considered as an endlink. Thus, a natural question would be: for any given point x in M, is there a way to construct a crooked sequence of chains, whose intersection is M, such that x is in an endlink of each chain? Theorem 3.3 is the first step to solving this problem. We will show how, if given a chain $\mathcal{D}$ crooked in $\mathcal{E}$, we can consolidate $\mathcal{D}$ in such a way that any particular link of $\mathcal{D}$ that we pick will be contained in an endlink of the consolidation.

This might not seem too amazing, since, if we let the diameter of the consolidated links be unlimited, we can simply say that the first link of the consolidation is the union of all links of the subchain from D_1 to the link we picked. As we will show, however, the consolidation can be made in such a way that each link of the consolidation is contained in at most 2 links of $\mathcal{E}$. Thus, the consolidated links are relatively small.

DEFINITION: Let $\mathcal{D} < \mathcal{E}$. $\mathcal{F}$ is a **limited consolidation** of $\mathcal{D}$ with respect to $\mathcal{E}$ if $\mathcal{F} \overset{>}{c} \mathcal{D}$ and each link of $\mathcal{F}$ is contained in at most 2 links of $\mathcal{E}$.

When doing a limited consolidation the size of the resulting links can be precisely controlled. For example, let $\mathcal{D} < \mathcal{E}$ such that each link of $\mathcal{E}$ has diameter less that 1/4. If $\mathcal{F}$ is a limited consolidation of $\mathcal{D}$ with respect to $\mathcal{E}$, then, since each link of $\mathcal{F}$ is contained in at most 2 links of $\mathcal{E}$, the links of $\mathcal{F}$ have diameter less than 1/2.

THEOREM 3.3 If $\mathcal{D}$ and $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ are chains, such that $\mathcal{D} \overset{<}{\approx} \mathcal{E}$, and if D_j is a particular link of $\mathcal{D}$, then there exists a limited consolidation $\mathcal{F}$ of $\mathcal{D}$ with respect to $\mathcal{E}$ such that D_j is contained only in F_1. Furthermore, any link of $\mathcal{F}$ that contains a link of $\mathcal{D}$ which intersects E_1 or E_n is contained in $E_1 \cup E_2$ or $E_{n-1} \cup E_n$, respectively.

Proof: If $n \leq 4$, then F_1 is simply the links of $\mathcal{D}$ contained in either the first two links of $\mathcal{E}$ or the last two links of $\mathcal{E}$, whichever contains D_j. F_2 would be all other links of $\mathcal{D}$ (if n = 1 or 2, there would be no F_2). So we will assume $n > 4$.

If $D_j \subset E_1 \cup E_2$, then take $\mathcal{F} = \{F_1, F_2, \ldots, F_{n-2}\}$ as the following and the theorem is satisfied:

$F_1 = \bigcup\{D \in \mathcal{D} \mid D \subset E_1 \cup E_2\}$,

$F_2 = \bigcup\{D \in \mathcal{D} \mid D \subset E_3 \text{ and } D \not\subset E_2\}$,

$F_3 = \bigcup\{D \in \mathcal{D} \mid D \subset E_4 \text{ and } D \not\subset E_3\}$,

:

$F_{n-1} = \bigcup\{D \in \mathcal{D} \mid D \subset E_{n-1} \cup E_n\}$.

If $D_j \subset E_{n-1} \cup E_n$, then $\mathcal{F}$ can be found in a similar manner.

For the remainder of the proof we will assume $D_j \subset \mathcal{E}(3, n\text{-}2)$ and use induction on n.

Suppose the theorem holds for $n \leq r\text{-}1$, for some integer r. We will show that it holds for $n = r$.

Let $\mathcal{D}(h, k)$ be the longest subchain of $\mathcal{D}$ containing D_j such that no interior link of $\mathcal{D}(h, k)$ intersects E_1 or E_r. Since $\mathcal{D}(h, k)$ is the longest subchain that meets the criteria, the endlinks of $\mathcal{D}(h, k)$ must intersect E_1 or E_r. There are two cases to consider.

<u>CASE 1</u>: Suppose both endlinks of $\mathcal{D}(h, k)$ intersect either E_1 or E_r, then $\mathcal{D}(h, k)$ is contained in r-1 or fewer links of $\mathcal{E}$. Thus, there is a limited consolidation, $\mathcal{H}$, of $\mathcal{D}(h, k)$ with respect to $\mathcal{E}$ such that $D_j \subset H_1$, but $D_j \not\subset H_2$.

<u>CASE 2</u>: Suppose one end link of $\mathcal{D}(h, k)$ intersects E_1 and the other intersects E_r. Without loss of generality, suppose $D_h \cap E_1 \neq \emptyset$ and $D_k \cap E_r \neq \emptyset$. Since $\mathcal{D} \leq \mathcal{E}$, there are links $D_s \subset E_{r-1}$ and $D_t \subset E_2$ such that $\mathcal{D}(h, k) = \mathcal{D}(h, s) \cup \mathcal{D}(s, t) \cup \mathcal{D}(t, k)$. Since no interior links of $\mathcal{D}(h, k)$ intersect E_1 or E_r, each of the subchains $\mathcal{D}(h, s)$, $\mathcal{D}(s, t)$, and $\mathcal{D}(t, k)$ are contained in r -1 or fewer links of $\mathcal{E}$.

Suppose first that $D_j \in \mathcal{D}(h, s)$. Since $\mathcal{D}(h, s)$ is contained in r -1 or fewer links, there is a limited consolidation $\mathcal{G}$ of $\mathcal{D}(h, s)$ with respect to $\mathcal{E}$ such that $D_j \subset G_1$. Let G_v be the first link of $\mathcal{G}$ that contains D_s. $D_s \subset E_{r-1}$ so $G_v \subset (E_{r-2} \cup E_{r-1})$. We can now use the chain $\mathcal{G}$ to construct a chain $\mathcal{H} = \{H_1, H_2, \ldots, H_{v+r+k-t-3}\}$, which is a limited consolidation of $\mathcal{D}(h, k)$ with respect to $\mathcal{E}$, such that $D_j \subset H_1$, as follows:

$\mathcal{H}(1, v\text{-}1) = \mathcal{G}(1, v\text{-}1)$,

$H_v = \bigcup\{D \in \mathcal{D} \mid D \subset (E_{r-2} \cup E_{r-1})$, but $D \not\subset (\bigcup \mathcal{G}(1, v\text{-}1)) \cup (\bigcup \mathcal{D}(t, k))\}$,

$H_{v+1} = \bigcup\{D \in \mathcal{D} \mid D \subset E_{r-3}$, but $D \not\subset (\bigcup \mathcal{G}(1, v\text{-}1)) \cup (\bigcup \mathcal{D}(t, k)) \cup E_{r-2}\}$,

$\vdots$

$H_{v+r-4} = \bigcup\{ D \in \mathcal{D} \mid D \subset E_2$, but $D \not\subset (\bigcup \mathcal{G}(1, v\text{-}1)) \cup (\bigcup \mathcal{D}(t, k)) \cup E_3\}$,

$\mathcal{H}(v + r - 3, v + r - 3 + k - t) = \mathcal{D}(t, k)$.

In a similar fashion we can find limited consolidations $\mathcal{H}$ if $D_j \in \mathcal{D}(s, t)$ or $D_j \in \mathcal{D}(t, k)$. Using the chain $\mathcal{H}$ found in each case, we can construct a limited consolidation $\mathcal{F}$ of $\mathcal{D}$ with respect to $\mathcal{E}$. Let H_u be the first link of $\mathcal{H}$ such that $H_u \cap (E_1 \cup E_r) \neq \emptyset$. Without loss of generality, suppose $H_u \cap E_1 \neq \emptyset$, then $H_u \subset E_1 \cup E_2$ and $\mathcal{F} = \{F_1, F_2, \ldots, F_{u+r-2}\}$ is constructed as follows:

$\mathcal{F}(1, u\text{-}1) = \mathcal{H}(1, u\text{-}1)$,

$F_u = \bigcup\{ D \in \mathcal{D} \mid D \subset E_1 \cup E_2$, but $D \not\subset (\bigcup \mathcal{H}(1, u\text{-}1))\}$,

$F_{u+1} = \bigcup\{ D \in \mathcal{D} \mid D \subset E_3$, but $D \not\subset E_2 \cup (\bigcup \mathcal{H}(1, u\text{-}1))\}$,

$\vdots$

$F_{u+r-1} = \bigcup\{ D \in \mathcal{D} \mid D \subset E_{r-2}$, but $D \not\subset E_{r-1} \cup (\bigcup \mathcal{H}(i, u\text{-}1))\}$,

$F_{u+r-2} = \bigcup\{ D \in \mathcal{D} \mid D \subset E_{r-1} \cup E_r$, but $D \not\subset (\bigcup \mathcal{H}(1, u - 1))\}$. ∎

Ultimately, we would like to form a crooked sequence of chains between two points x and y of M. If the intersection of this crooked sequence of chains equaled M, then x and y would be endpoints. Unfortunately using the last theorem we can only be sure that x is in F_1, but we need y to be in the last link of $\mathcal{F}$ as well. We will show that this can be done in certain cases. Specifically, if D_r and D_s are links of $\mathcal{D} \lesssim \mathcal{E}$ and the subchain $\mathcal{D}(r, s)$ meets both the first and last links of $\mathcal{E}$, then the consolidation $\mathcal{F}$ of $\mathcal{D}$ can be made so that D_r and D_s are in opposite endlinks. We will see in Chapter 4 that this requirement is met when x and y are in different composants. Again, this is a limited consolidation of $\mathcal{D}$ with respect to $\mathcal{E}$.

THEOREM 3.4 If a chain $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ is crooked in a chain $\mathcal{E} = \{E_1, E_2, \ldots, E_m\}$ and if $\mathcal{D}(r, s)$ is a subchain of $\mathcal{D}$ that has links intersecting E_1 and E_m, then there is a limited consolidation $\mathcal{F}$ of $\mathcal{D}$ such that $D_r \subset F_1$ and $D_s \subset F_L$, where F_1 and F_L are end links of $\mathcal{F}$.

Proof: Let $\mathcal{D}(h, k)$ be a subchain of $\mathcal{D}(r,s)$ such that $D_h \cap E_1 \neq \emptyset \neq D_k \cap E_m$. Without loss of generality, suppose $1 \leq r \leq h \leq k \leq s \leq n$.

By Theorem 3.3, there is a limited consolidation $\mathcal{H}$ of $\mathcal{D}(1, h)$ such that $D_r \subset H_1$, and there is a limited consolidation $\mathcal{G}$ of $D(k, n)$ such that $D_s \subset G_1$.

Let H_u be the first link of $\mathcal{H}$ containing D_h and let G_v be the first link of $\mathcal{G}$ containing D_k. Then, $H_u \subset E_1 \cup E_2$, $G_v \subset E_{m-1} \cup E_m$, and $\mathcal{F} = \{F_1, F_2, \ldots, F_{u+m+v-3}\}$ is constructed as follows:

$\mathcal{F}(1, u - 1) = \mathcal{H}(1, u - 1)$,

$F_u = \bigcup\{\, D \in \mathcal{D} \mid D \subset E_1 \cup E_2, \text{but } D \not\subset (\bigcup\mathcal{H}(1, u - 1)) \cup (\bigcup\mathcal{G}(1, v - 1))\}$,

$F_{u+1} = \bigcup\{\, D \in \mathcal{D} \mid D \subset E_3, \text{but } D \not\subset (\bigcup\mathcal{H}(1, u - 1)) \cup (\bigcup\mathcal{G}(1, v - 1)) \cup E_2\}$,

$\vdots$

$F_{u+m-4} = \bigcup\{\, D \in \mathcal{D} \mid D \subset E_{m-2}, \text{but } D \not\subset (\bigcup\mathcal{H}(1, u - 1)) \cup (\bigcup\mathcal{G}(1, v - 1))$
$\cup\, E_{m-3}\}$,

$F_{u+m-3} = \bigcup\{\, D \in \mathcal{D} \mid D \subset E_{m-1} \cup E_m, \text{but } D \not\subset (\bigcup\mathcal{H}(1, u - 1)) \cup (\bigcup\mathcal{G}(1, v - 1))$
$\cup\, E_{m-2}\}$,

$\mathcal{F}(u+m-2, u+m+v-3) = \mathcal{G}(v - 1, 1)$. ■

At this point, given $\mathcal{D} \precsim \mathcal{E}$, we would like to consolidate $\mathcal{D}$ in such a way that the consolidation follows some given pattern in $\mathcal{E}$. Unfortunatly, this can not always be done. For example, if $\mathcal{D}$ is a maximal crooked chain with respect to $\mathcal{E}$, as described in Appendix B, then any consolidation of $\mathcal{D}$ results in a chain that is no longer a refinement of $\mathcal{E}$. However, if instead we are given a crooked sequence of chains $\mathcal{D}_1, \mathcal{D}_2, \ldots$ from p to q, then for any given pattern there is an integer j such that $\mathcal{D}_j$ can be consolidated so as to follow the given pattern in $\mathcal{D}_1$. This fact will follow from the next theorem.

LEMMA 3.5: Let $\mathcal{D}_1, \mathcal{D}_2, \ldots$ be a crooked sequence of chains from p to q. Suppose $\mathcal{D}_2(u, v) < \mathcal{D}_1(h, k)$ is such that $Cl(D(2)_u) \cap Cl(D(2)_v) = \emptyset$, $Cl(D(2)_u) \subset D(1)_h$, and $Cl(D(2)_v) \subset D(1)_k$. Then for every integer λ there exists an integer $m > \lambda$ and some intgers i and j such that $D(m)_i \subset D(1)_h \cap (\bigcup \mathcal{D}_2(u,v))$, but $D(m)_i \cap D(2)_u = \emptyset$ and $D(m)_j \subset D(1)_k \cap (\bigcup \mathcal{D}_2(u,v))$, but $D(m)_j \cap D(2)_v = \emptyset$.

Proof: Let λ be given.

Since $Cl(D(2)_u) \subset D(1)_h$ and $Cl(D(2)_v) \subset D(1)_k$, there exists an $\varepsilon > 0$ such that $N_\varepsilon(D(2)_u) \subset D(1)_h$ and $N_\varepsilon(D(2)_v) \subset D(1)_k$. This means that any set with diameter less than ε that meets $D(2)_u$ or $D(2)_v$ is contained in $D(1)_h$ or $D(1)_k$, respectively.

Let m be an integer such that $m > \max\{\lambda, 3/\varepsilon\}$.

Suppose $\mathcal{D}_m(i\text{-}2, i) < \mathcal{D}_2(u, v)$ is a 3 linked subchain such that only $D(m)_{i-2}$ meets $D(2)_u$. Then $D(m)_i \cap D(2)_u = \emptyset$. But, $\operatorname{diam}(\bigcup \mathcal{D}_m(i\text{-}2, i)) < 3(1/m) < 3(\varepsilon/3) = \varepsilon$, so, $D(m)_i \subset D(1)_h \cap (\bigcup \mathcal{D}_2(u,v))$. The same argument can be used to show that there exists j such that $D(m)_j \subset D(1)_k \cap (\bigcup \mathcal{D}_2(u,v))$ but $D(m)_j \cap D(2)_v = \emptyset$. ■

THEOREM 3.6: Let $\mathcal{D}_1, \mathcal{D}_2, \ldots$ be a crooked sequence of chains from p to q. Suppose $\mathcal{D}_2(u, v) < \mathcal{D}_1(h, k)$ is such that $Cl(D(2)_u) \cap Cl(D(2)_v) = \emptyset$, $Cl(D(2)_u) \subset D(1)_h$, $Cl(D(2)_v) \subset D(1)_k$ and $f:\{1, 2, \ldots, n\} \to \mathbb{N}$ is a pattern such that $h = f(1) \leq f(i) \leq f(n) = k$, for every $i = 2 \ldots n-1$. Then for every integer λ, there exists $j > \lambda$ and a chain $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ which is a consolidation of the links of $\mathcal{D}_j$ contained in $\mathcal{D}_2(u, v)$ such that:

i) $\mathcal{E}$ follows the pattern f in $\mathcal{D}_1(h, k)$, and

ii) no interior link of $\mathcal{E}$ intersects $D(2)_u$ or $D(2)_v$.

Proof: (by induction on n, the length of the chain $\mathcal{E}$)

[Remark: It is important that j be picked to be large enough so that there is at least one link of $\mathcal{D}_j$ contained in $D(1)_h$ that does not meet $D(2)_u$. If this were not the case then some interior link of $\mathcal{E}$ would have to intersect $D(2)_u$. That such a j can always be found follows from Lemma 3.5. (Likewise, j must also be large enough so that there is at least one link of $\mathcal{D}_j$ contained in $D(1)_k$ that does not meet $D(2)_v$).]

For every $j > 2$, let $\mathcal{D}_j^*$ denote the links of $\mathcal{D}_j$ contained in $\mathcal{D}_2(u, v)$. Note that the links of $\mathcal{D}_j^*$ may not form a subchain of $\mathcal{D}_j$.

Let an integer λ be given. From Lemma 3.5 there exists an integer $m > \lambda$ such that some link of $\mathcal{D}_m$ is contained in $D(1)_h \cap (\bigcup \mathcal{D}_2(u,v))$, but does not meet $D(2)_u$ and some link of $\mathcal{D}_m$ is contained in $D(1)_k \cap (\bigcup \mathcal{D}_2(u,v))$, but does not meet $D(2)_v$. For the remainder of the proof we will pick $j \geq m$, which, of course, means $j > \lambda$.

$D_1(h, k)$ has $k - h + 1$ links and $|f(i) - f(i + 1)| \leq 1$ so $n \geq k - h + 1$. ($n < k - h + 1$ would imply $|f(i) - f(i + 1)| > 1$ for some i).

If $n = k - h + 1$ then the pattern $f(i) = h + i - 1$, for every $i = 1$ to n. For this case, let j be any integer greater than m and construct $\mathcal{E}$ as follows:

$E_1 = \bigcup\{D \in \mathcal{D}_j^* \mid D \subset D(1)_h\}$,

$E_2 = \bigcup\{D \in \mathcal{D}_j^* \mid D \subset D(1)_{h+1}$, but $D \not\subset D(i)_h\}$,

$\vdots$

$E_n = \bigcup\{D \in \mathcal{D}_j^* \mid D \subset D(1)_k\}$.

Since D_h and D_k are distinct links, $k - h + 1 \geq 2$. In particular, if $n = 2$, then $k - h + 1 = 2 = n$ and the theorem holds.

Let $a \geq 3$ and suppose the theorem holds for all $n \leq a - 1$ and for all possible choices of h and k such that $n \geq k - h + 1$ and $k > h \geq 1$. We will show the theorem holds for $n = a$.

Let $n = a$ and consider the following 3 cases (each case represents different patterns $\mathcal{E}$ could follow).

CASE 1: Suppose $f(1) = f(2) = h$ (the first two links of $\mathcal{E}$ are to be contained in $D(1)_h$).

Since the theorem holds for $n = a - 1$, there exists an integer $j > m$ and a chain $\mathcal{F} = \{F_2, F_3, \ldots, F_a\}$ which is a consolidation of the links of $\mathcal{D}_j^*$ such that $\mathcal{F}$ follows the pattern $f(2), f(3), \ldots, f(a)$ in $\mathcal{D}_1$, and no interior link of $\mathcal{F}$ intersects $D(2)_u$ or $D(2)_v$.

Since $f(1) = f(2)$, $\mathcal{E} = \{E_1, E_2, \ldots, E_a\}$ can be constructed as follows:

$E_1 = \bigcup\{D \in \mathcal{D}_j^* \mid D \subset F_2$ and $D \cap D(2)_u \neq \emptyset\}$,

$E_2 = \bigcup\{D \in \mathcal{D}_j^* \mid D \subset F_2$ and $D \cap D(2)_u = \emptyset\}$,

$E_3 = F_3$,

$\vdots$

$E_a = F_a$.

Note that $E_2 \neq \emptyset$ since $j > m$.

For the remainder of the cases we will assume that $f(1) \neq f(2)$.

CASE 2: Suppose $f(i) \neq f(1)$ for every $i > 1$ (only the first link of $\mathcal{E}$ is to meet

$D(1)_h$).

Let $\mathcal{W} = \{D \in \mathcal{D}_2(u, v) \mid D \not\subset D(1)_h\}$ and let $\mathcal{X} = \{D \in \mathcal{W} \mid D \cap D(1)_h \neq \emptyset\}$. By the inductive hypothesis, there exists an integer $j > m$ such that $\mathcal{F} = \{F_2, F_3, \ldots, F_a\}$ is a consolidation of the links of $\mathcal{D}_j{}^*$ contained in $\mathcal{W}$ that follows the pattern f(2), f(3), . . . , f(a) in $\mathcal{D}_1$, only F_2 meets $\cup \mathcal{X}$, and only F_a meets $D(2)_v$. $\mathcal{E}$ is defined as follows:

$$
\begin{aligned}
E_1 &= \cup\{\, D \in \mathcal{D}_j{}^* \mid D \subset D(1)_h\},\\
E_2 &= F_2,\\
E_3 &= F_3,\\
&\vdots\\
E_n &= F_a.
\end{aligned}
$$

<u>CASE 3</u>: Suppose f(1) = f(i) for some i. Since f(1) $\neq$ f(2), there exists integers r and t such that $1 < t < r < a$, f(1) = f(r) < f(t), and f(i) $\leq$ f(t) for every i = 1, 2, . . ., r (the chain $\mathcal{E}$ leaves $D(1)_h$, reaches some maximal distance, and then return to $D(1)_h$. Figure 3-3 shows how this pattern might look.)

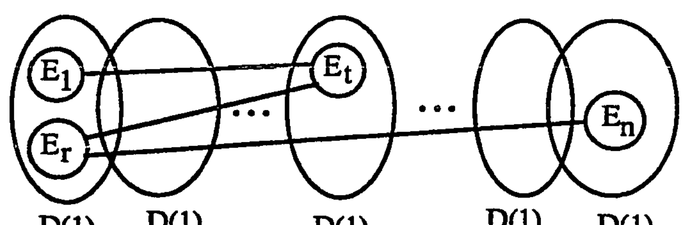


Figure 3-3

[To prove this case we will do three separate consolidations following the patterns {f(1), f(2), . . ,f(t)}, {f(t), f(t+1), . . . , f(r)}, and {f(r). f(r+1), . . . , f(a)}, respectively.

Using the inductive hypothesis, all three of these consolidations can be done since each has less than a links. Once a large enough integer j is found, so that all 3 consolidations can be done, the three chains will be joined together at the obvious overlaps and the consolidation $\mathcal{C}$ will have been found. First, we must find a way to group the links that will eventually be consolidated to follow those three patterns.]

Suppose $\mathcal{D}_{m+1}(w, z) < \mathcal{D}_2(u, v)$ such that $D(m+1)_w \cap D(2)_u \neq \emptyset$, and $D(m+1)_z \cap D(1)_{f(t)+1} \neq \emptyset$. Since $\mathcal{D}_{m+1} \lesssim \mathcal{D}_m$, there exists b and c such that $\mathcal{D}_{m+1}(w, z) \lesssim \mathcal{D}_m(b, c) < \mathcal{D}_1(h, k)$. Hence, there are integers x and y such that $w < x < y < z$ and $\mathcal{D}_{m+1}(w, z) = \mathcal{D}_{m+1}(w, x) \cup \mathcal{D}_{m+1}(x, y) \cup \mathcal{D}_{m+1}(y, z)$, where $\mathcal{D}_{m+1}(w, x) < \mathcal{D}_1(h, f(t))$, $\mathcal{D}_{m+1}(w, y) \cap D(2)_v = \emptyset$, $Cl(D(m+1)_x) \subset D(1)_{f(t)}$, and $Cl(D(m+1)_y) \subset D(1)_h$.

We can now group the links of $\mathcal{D}_{m+1}{}^*$ into 5 subcollections $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, and $\mathcal{V}$. The links contained in these subcollections will be used to form three chains following the patterns in the above remark. The subchain $\mathcal{D}_{m+1}(w, z)$, from the above paragraph, will be used to show that none of these subcollections is empty. Let $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, and $\mathcal{V}$ meet the following criteria:

i) $\{D \in \mathcal{D}_{m+1}{}^* \mid D \subset D(2)_u\} \subset \mathcal{R}$,

ii) $\mathcal{U} = \{D \in \mathcal{D}_{m+1}{}^* \mid D \subset D(1)_{f(t)}\}$,

iii) $\mathcal{V} = \{D \in \mathcal{D}_{m+1}{}^* \mid D \subset D(1)_h\}$,

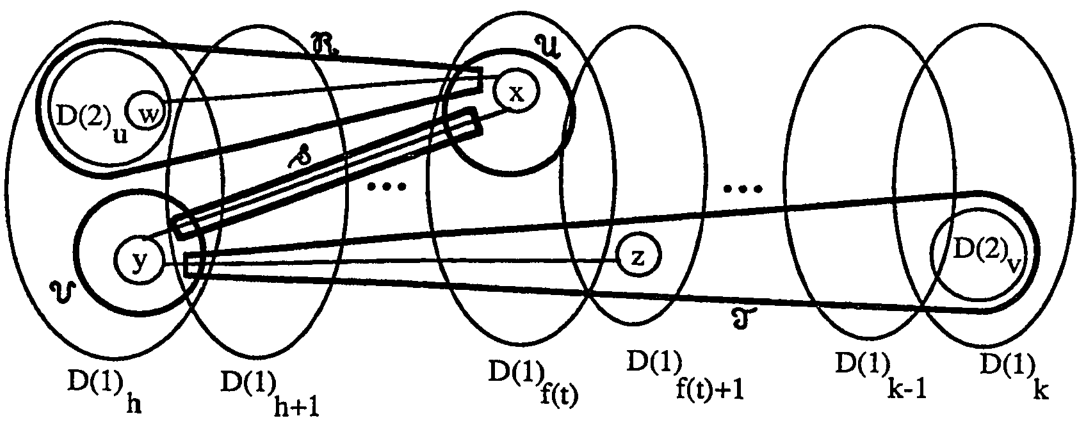
iv) $\{D \in \mathcal{D}_{m+1}{}^* \mid D \subset D(2)_v\} \subset \mathcal{T}$,

v) $(\mathcal{R} \cup \mathcal{U}) \cap (\mathcal{V} \cup \mathcal{T}) = \emptyset$, and

vi) each link of $(\mathcal{R} \cup \mathcal{U} \cup \mathcal{V} \cup \mathcal{S})$ is contained in some link of $\mathcal{D}_1(h, f(t))$.

From these criteria we can see that $\mathcal{R}$ partly consists of the subchains of $\mathcal{D}_{m+1}{}^*$ which are contained in $\mathcal{D}_1(h, f(t))$ and join $\mathcal{U}$ with $D(2)_u$ ($\mathcal{D}_{m+1}(w, x)$ for example). $\mathcal{S}$ consists of the subchains of $\mathcal{D}_{m+1}{}^*$ which are contained in $\mathcal{D}_1(h, f(t))$ and join $\mathcal{U}$ with $\mathcal{V}$ ($\mathcal{D}_{m+1}(x, y)$ for example). $\mathcal{T}$ partly consists of the subchains of $\mathcal{D}_{m+1}{}^*$ that join $\mathcal{V}$ with $D(2)_v$ ($\mathcal{D}_{m+1}(y, z) \subset \mathcal{T}$ for example).

Figure 3-4 shows an example of how the collections might meet the criteria.



The letters w, x, y, and z represent links the links $D(m+1)_w$, $D(m+1)_w$, $D(m+1)_w$, and $D(m+1)_w$, respectively.

Figure 3-4

Since the theorem holds for $n \leq a-1$, we can do the following three consolidations:

1) There exists an integer $\alpha > m$ such that $\mathcal{F} = \{F_1, \ldots, F_t\} \overset{>}{c} \{ D \in \mathcal{D}_\alpha{}^* \mid D \subset (\mathcal{R} \cup \mathcal{U})\}$, $\mathcal{F}$ follows the pattern f(1), f(2), . . . , f(t) in $\mathcal{D}_1$, and no interior links of $\mathcal{F}$ meet $D(2)_u$ or $\cup \mathcal{U}$.

2) There exists an integer $\beta > m$ such that $\mathcal{G} = \{G_t, \ldots, G_r\} \overset{>}{c} \{ D \in \mathcal{D}_\beta{}^* \mid D \subset (\mathcal{U} \cup \mathcal{S} \cup \mathcal{V})\}$, $\mathcal{G}$ follows the pattern f(t), f(t+1), . . . , f(r) in $\mathcal{D}_1$, and no interior

links of $\mathcal{G}$ meet $\cup\mathcal{U}$ or $\cup\mathcal{V}$.

3) There exists $\gamma > m$ such that $\mathcal{H} = \{H_r, \ldots, H_a\} \overset{>}{c} \{ D \in \mathcal{D}_\gamma^* \mid D \subset (\mathcal{V} \cup \mathcal{T})\}$, $\mathcal{H}$ follows the pattern f(r), f(r+1), . . , f(a) in $\mathcal{D}_1$, and no interior link of $\mathcal{G}$ meets $\cup\mathcal{V}$ or $D(2)_v$.

Using these three consolidations, the integer j and the chain $\mathcal{E} = \{E_1, \ldots, E_t, \ldots, E_r, \ldots, E_a\}$ can be found as follows: Let $j = \max\{\alpha, \beta, \gamma\}$

$E_1 = \cup\{ D \in \mathcal{D}_j^* \mid D \subset (D(2)_u \cup F_1)\}$,

$E_i = \cup\{ D \in \mathcal{D}_j^* \mid D \subset F_i \}$ for i = 2, 3, . . , t - 1,

$E_t = \cup\{ D \in \mathcal{D}_j^* \mid D \subset F_t \cup G_t\}$,

$E_i = \cup\{ D \in \mathcal{D}_j^* \mid D \subset G_i \}$ for i = t+1, t+2, . . , r-1,

$E_r = \cup\{ D \in \mathcal{D}_j^* \mid D \subset G_r \cup H_1\}$,

$E_i = \cup\{ D \in \mathcal{D}_j^* \mid D \subset H_i \}$ for i = r+1, r+2, . . a-1, and

$E_a = \cup\{ D \in \mathcal{D}_j^* \mid D \subset (D(2)_v \cup H_a)\}$.

The three cases have exhausted the possible patterns and for each case we have found a chain $\mathcal{E}$ that meets the requirements. Therefore the theorem is proved. ■

By taking the subchain $\mathcal{D}_2(u,v)$ to be all of $\mathcal{D}_2$ we get the following theorem.

Theorem 3.7: Suppose $\mathcal{D}_1, \mathcal{D}_2, \ldots$ is a crooked sequence of chains from p to q and $f:\{1, \ldots, n\} \to \mathbb{N}$ is a pattern such that $1 = f(1) \le f(i) \le f(n)$ for every i = 2, . . , n-1. Then there exists an integer j and a chain $\mathcal{E}$ from p to q which is a consolidation of $\mathcal{D}_j$ such thtat $\mathcal{E}$ follows the pattern f in $\mathcal{D}_1$.

CHAPTER 4 Some Properties of the Pseudo Arc

In this chapter we will prove that the pseudo arc is homogeneous. In Chapter 1 a brief description of homogeneity was given; here a formal definition is presented.

DEFINITION: A continuum X is **homogeneous** if for every pair of points p and q of X there exists a homeomorphism $g:X \to X$ such that $g(p) = q$.

To prove the pseudo arc is homogeneous we will need some more general theorems regarding homeomorphisms and chainable continua. As a result of Theorem 4.3, we will finally state that all pseudo arcs are homeomorphic, regardless of the crooked sequence used to construct them. First we must define chain cover and chainable continua.

DEFINITION: Let X be a continuum. $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ is a **chain cover** if $\mathcal{C}$ is a chain, C_i is open in X for every $i = 1, 2, \ldots, n$, and $\bigcup C_i = X$. A chain cover $\mathcal{C}$ is an **ε-chain cover** if $\operatorname{diam}(C_i) < \varepsilon$ for every $i = 1, 2, \ldots, n$.

DEFINITION: Let X be a continuum. X is **chainable** if for every $\varepsilon > 0$ there is an ε-chain cover of X.

Examples: An arc is certainly chainable, as is the sin 1/x curve and Knaster's bucket handles. A circle is not chainable, because there would always be the problem of the first and last links intersecting, and, hence, there would be no chain. A triod (three line segments radiating from a point) is also not chainable. If a small enough ε is picked there would be three subchains radiating from one link. Again, this is not a chain.

THEOREM 4.1: Let X, Y be chainable continua and $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers with a finite sum. Suppose $\mathcal{C}_i$ and $\mathcal{D}_i$ are chain covers of X and Y, respectively, such that:

i) each link of $\mathcal{C}_i$ and $\mathcal{D}_i$ has diameter less that ε_i,

ii) $\mathcal{C}_i$ and $\mathcal{D}_i$ have the same number of links, and

iii) if $C(i+1)_j \cap C(i)_k \neq \emptyset$, then $d(D(i+1)_j, D(i)_k) \leq \varepsilon_i$.

Then there exists a continuous function g from X onto Y.

Proof: The proof has four parts. We will first define the function g, then show that it is well defined, continuous, and onto.

Part 1: We define the function g as follows:

Let $E(i)_k = \{ q \in Y \mid d(q, D(i)_k) < \varepsilon_i + 2(\varepsilon_{i+1} + \varepsilon_{i+2} + \ldots)\}$

and let $\mathcal{E}_i$ be the well ordered collection whose k^{th} element is $E(i)_k$. Then $\mathcal{E}_i$ is a chain cover of Y. Note that diam $(E(i)_k) < 3\varepsilon_i + 4(\varepsilon_{i+1} + \varepsilon_{i+2} + \ldots)$. Hence, diam $(E(i)_k) \to 0$ as $i \to \infty$.

Claim: If $C(i)_j \cap C(i+1)_k \neq \emptyset$, then $Cl(E(i+1)_k) \subset E(i)_j$.

Let $y \in Cl(E(i+1)_k)$. Then $d(y, D(i+1)_k) \leq \varepsilon_{i+1} + 2(\varepsilon_{i+2} + \varepsilon_{i+3} + \ldots) < 2(\varepsilon_{i+1} + \varepsilon_{i+2} + \ldots)$. Since $C(i)_j \cap C(i+1)_k \neq \emptyset$, we have $d(D(i)_j, D(i+1)_k) < \varepsilon_i$. So, by the triangle inequality $d(y, D(i)_j) \leq d(y, D(i+1)_k) + d(D(i)_j, D(i+1)_k) < \varepsilon_i + 2(\varepsilon_{i+1} + \varepsilon_{i+2} + \ldots)$. Therefore, $y \in E(i)_j$ and the claim is proved.

Let $p \in X$ and let $C(1)_a, C(2)_b, \ldots$ be a sequence of links, one from each $\mathcal{C}_i$, that contains p. We define $g(p) = E(1)_a \cap E(2)_b \cap \ldots$

Note that $g(C(i)_k) \subset E(i)_k$ for every i and k.

Part 2: Let $\langle C(k)_{ak} \rangle$ and $\langle C(k)_{bk} \rangle$ be sequences such that, for every $k = 1, 2, \ldots$, $p \in C(k)_{ak} \cap C(k)_{bk}$ and $C(k)_{ak}, C(k)_{bk} \in \mathcal{C}_k$. Let $x_1 = \cap E(k)_{ak}$ and $x_2 = \cap E(k)_{bk}$. To show that g is well defined, we must show that $x_1 = x_2$.

On the contrary, suppose $x_1 \neq x_2$. Let $d(x_1, x_2) = \delta > 0$. Since the sequence $\varepsilon_1, \varepsilon_2, \ldots$ has a finite sum, there is an integer i such that $\varepsilon_i + 2(\varepsilon_{i+1} + \varepsilon_{i+2} + \ldots) < \delta/9$.

Let $r = a_i$ and $s = b_{i+1}$. Then $C(i)_r$ and $C(i+1)_s$ are elements of the first and second sequences, respectively, which implies $x_1 \in E(i)_r$ and $x_2 \in E(i+1)_s$. Since the diameters of $E(i)_r$ and $E(i+1)_s$ are both less than $\delta/3$ and since $d(x_1, x_2) = \delta$, $E(i)_r \cap E(i+1)_s = \emptyset$. But $p \in C(i)_r \cap C(i+1)_s$, so by the claim we have $E(i+1)_s \subset E(i)_r$, which is a contradiction. Therefore, $x_1 = x_2$ and the function g is well defined.

Part 3: Next the continuity of g will be shown. Let V be an open set containing g(p). Since the ε_i have a finite sum, $\varepsilon_i \to 0$ as $i \to \infty$. This implies that there exists an integer j such that if $g(p) \in E(j)_k$ for some k, then $E(j)_k \subset V$.

Let $p \in C(j)_r$, then $C(j)_r$ is an open set containing p. Then we have $g(C(j)_r) \subset E(j)_r \subset V$. Therefore, g is continuous.

Part 4: Finally we will show that g(X) is dense in Y. Let $y \in Y$ and let U be any open set in Y such that $y \in U$. We will show that $U \cap g(X) \neq \emptyset$.

There exists an integer j such that if $y \in E(j)_k$ for some k, then $E(j)_k \subset U$. Let $D(j)_r$ be a link of $\mathcal{D}_j$ that contains y, then $D(j)_r \subset E(j)_r \subset U$. By the way g was defined, $g(C(j)_r) \subset E(j)_r \subset U$, hence, $U \cap g(X) \neq \emptyset$. This implies that g(X) is dense in Y. But X is compact, so g(X) is closed. This implies $g(X) = Y$ or, in otherwords, g is onto.

Therefore, the desired function has been found. ■

If the requirement that $d(C(i+1)_j, C(i)_k) \leq \varepsilon_i$ if $D(i+1)_j \cap D(i)_k \neq \emptyset$ is added to the last theorem, then we can show X and Y are homeomorphic.

THEOREM 4.2: Let X, Y be chainable continua and $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers with a finite sum. Suppose $\mathcal{C}_i$ and $\mathcal{D}_i$ are chain covers of X and Y, respectively, such that:

i) each link of $\mathcal{C}_i$ and $\mathcal{D}_i$ has diameter less that ε_i,

ii) $\mathcal{C}_i$ and $\mathcal{D}_i$ have the same number of links,

iii) if $C(i+1)_j \cap C(i)_k \neq \emptyset$, then $d(D(i+1)_j, D(i)_k) \leq \varepsilon_i$, and

iv) if $D(i+1)_j \cap D(i)_k \neq \emptyset$, then $d(C(i+1)_j, C(i)_k) \leq \varepsilon_i$.

Then there exists a homeomorphism $g:X \to Y$.

Proof: Let $g:X \to Y$ be the continuous surjection constructed in the proof of Theorem 4.1. To show that g is a homeomorphism all that must be shown is that g is one to one.

Suppose this is not true, then there exists two points a and b in X such that $g(a) = g(b)$

Let $F(i)_k = \{ q \in X \mid d(q, C(i)_k) < \varepsilon_i + 2(\varepsilon_{i+1} + \varepsilon_{i+2} + \ldots)\}$

and let $\mathfrak{F}_i$ be the well ordered collection whose k^{th} element is $F(i)_k$. Then $\mathfrak{F}_i$ is a chain cover of X. By the same arguement used in theorem 4.1, if $D(i+1)_j \cap D(i)_k \neq \emptyset$, then $Cl(F(i+1)_k \subset F(i)_j$.

Since $a \neq b$, $d(a, b) = \varepsilon > 0$. This means that there exists an integer k such that $\varepsilon/9 > \varepsilon_k + 2(\varepsilon_{k+1} + \varepsilon_{k+2} + \ldots)$ and thus, no element of $\mathfrak{F}_k$ contains both a and b as in part 2 of the previous proof.

Suppose $y \in D(k)_r$, then there is an integer $j > k$ such that if $y \in E(j)_i$ for some i, then $E(j)_i \subset D(k)_r$. ($E(j)_i$ was defined in the proof of theorem 4.1) Let $C(j)_u$ and $C(j)_v$ be elements of $\mathfrak{C}_j$ containing a and b respectively. Then by the definition of g, $g(a) \in E(j)_u \subset D(k)_r$ and $g(b) \in E(j)_v \subset D(k)_r$. But $D(j)_u \subset E(j)_u$ and $D(j)_v \subset E(j)_v$, so both $D(j)_u$ and $D(j)_v$ are contained in $D(k)_r$.

Since $D(j)_u \cap D(k)_r \neq \emptyset$, $Cl(F(j)_u) \subset F(k)_r$. Likewise $D(j)_v \cap D(k)_r \neq \emptyset$ implies $Cl(F(j)_v) \subset F(k)_r$. However, this implies that $a \in C(j)_u \subset F(j)_u \subset F(k)_r$ and $b \in C(j)_v \subset F(j)_v \subset F(k)_r$. In other words, both a and b are contained in $F(k)_r$, which contradicts the fact that no element of $\mathfrak{F}_k$ contains a and b. Thus g is one to one.

By the above result and theorem 4.1, g is a homeomorphism. ∎

It is clear that the pseudo arc is chainable. After all, the construction we described required the use of chains. From Theorem 4.2 it would seem that we could easily say that two pseudo arcs are homeomorphic. However, there is a problem with this conclusion; the chains used to construct the pseudo arc do not meet the requirements of the theorem. Specifically, the diameter of the links of successive chains do not have a finite sum and the chain $\mathcal{D}_i$ covering one pseudo arc may not have the same number of links as the chain $\mathcal{C}_i$ covering another, although, by using the properties of crooked chains, both of these problems can be overcome. Suppose, for example, we to use only the chains corresponding to $i = 2, 4, 8, \ldots$, then the sum of the diameters would converge. Unfortunately, it turns out that the solution is not so simple, because we would still have the problem of the number of links not being equal. In the following theorem we will use consolidations to force the chains to have the same number of links and keep the diameters of the links small.

THEOREM 4.3: Suppose $\mathcal{C}_1, \mathcal{C}_2, \ldots$ is a crooked sequence of chains from p_1 to q_1 and $\mathcal{D}_1, \mathcal{D}_2, \ldots$ is a crooked sequence of chains from p_2 to q_2. Let $M_1 = \bigcap(\mathrm{Cl}(\bigcup \mathcal{C}_i))$ and $M_2 = \bigcap(\mathrm{Cl}(\bigcup \mathcal{D}_i))$. Then there is a homeomorphism $g{:}M_1 \to M_2$ such that $g(p_1) = p_2$ and $g(q_1) = q_2$.

Proof: We want to find a sequence of chain covers $\mathcal{A}_1, \mathcal{A}_2, \ldots$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots$ of M_1 and M_2, respectively, which meet the requirements of theorem 4.2.

Recall that a crooked sequence of chains has the property that the diameters of the links of $\mathcal{C}_i < 1/i$.

Let $\mathcal{A}_1 = \mathcal{C}_2$. (The diameter of links of $\mathcal{A}_1$ is less than 1/2.)

Let $\mathcal{B}_1$ be a chain from p_2 to q_2 which is a consolidation of some $\mathcal{D}_j$ such that $\mathcal{B}_1$ has the same number of links as $\mathcal{A}_1$.(Note that we do not know the diameters of the links of $\mathcal{B}_1$.)

Since $\mathcal{B}_1 \underset{c}{\geq} \mathcal{D}_j$ there exists an integer k such that $\mathcal{D}_k < \mathcal{B}_1$ and $k \geq 2$. Let $\mathcal{B}_2 = \mathcal{D}_k$. (The diameter of links of $\mathcal{B}_2$ is less thatn 1/2.)

By Theorem 3.7 there is an integer j such that $\mathcal{A}_2 \underset{c}{\geq} \mathcal{C}_j$ and $\mathcal{A}_2$ is a chain from p_1 to q_1 that follows the same pattern in $\mathcal{A}_1$ that $\mathcal{B}_2$ follows in $\mathcal{B}_1$.

Let $\mathcal{A}_3 = \mathcal{C}_i$ such that $i > j$ and $i \geq 4$. (The diameter of links of $\mathcal{A}_3$ is less than 1/4.)

Again, by Theorem 3.7 there is an integer j such that $\mathcal{B}_3 \underset{c}{\geq} \mathcal{D}_j$ and $\mathcal{B}_3$ is a chain from p_2 to q_2 that follows the same pattern in $\mathcal{B}_2$ that $\mathcal{A}_3$ follows in $\mathcal{A}_2$.

By continuing this process we get a sequence of chain covers $\mathcal{A}_1, \mathcal{A}_2, ..$ and $\mathcal{B}_1, \mathcal{B}_2, ..$ covering M_1 and M_2 respectively. Figure 4-1 shows the order in which the chains are found. We know the links of $\mathcal{A}_1$ and $\mathcal{B}_2$ have diameters less thatn 1/2, but $\mathcal{A}_2 < \mathcal{A}_1$ and $\mathcal{B}_3 < \mathcal{B}_2$ so the links of both $\mathcal{A}_2$ and $\mathcal{B}_3$ must also have diameter less than 1/2. The links of $\mathcal{A}_3$ were picked so as to have diameters less than 1/4, which is less than 1/2. The same reasoning can be used to show that the links of $\mathcal{A}_{2i}$, $\mathcal{A}_{2i+1}$, $\mathcal{B}_{2i}$, and $\mathcal{B}_{2i+1}$ have diameters less than $1/(2^i)$.

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d] & \mathcal{A}_2 \arrow[r] & \mathcal{A}_3 \arrow[d] & \\
\mathcal{B}_1 \arrow[r] & \mathcal{B}_2 \arrow[u] & \mathcal{B}_3 & \cdots
\end{tikzcd}$$

Figure 4-1

Let $\varepsilon_{2k} = 1/(2^k)$ and $\varepsilon_{2k+1} = 1/(2^k)$ for every integer $k \geq 1$. Then $\Sigma\varepsilon_i = 2$ and each link of $\mathcal{A}_i$ and each link of $\mathcal{B}_i$ has diameter less that ε_i for every integer $i \geq 2$ (this is only true for $i \geq 2$ since we do not know the diameter of $\mathcal{B}_1$).

We also see that $\mathcal{A}_{i+1}$ follows the same pattern in $\mathcal{A}_i$ that $\mathcal{B}_{i+1}$ follows in $\mathcal{B}_i$.

Hence, each chain $\mathcal{A}_i$ has the same number of links as $\mathcal{B}_i$. All that remains to be shown is that if $A(i)_j \cap A(i+1)_k \neq \emptyset$, then $d(B(i)_j, B(i+1)_k) < \varepsilon_i$ (likewise, if $B(i)_j \cap B(i+1)_k \neq \emptyset$, then $d(A(i)_j, A(i+1)_k) < \varepsilon_i$)

Suppose $A(i)_j \cap A(i+1)_k \neq \emptyset$. Since $\mathcal{A}_{i+1} < \mathcal{A}_i$, $A(i+1)_k$ is contained in $A(i)_{j-1}$ or $A(i)_j$ or $A(i)_{j+1}$. Since $\mathcal{B}_{i+1}$ follows the same pattern in $\mathcal{B}_i$ that $\mathcal{A}_{i+1}$ follows in $\mathcal{A}_i$, $B(i+1)_k$ must be contained in $B(i)_{j-1}$ or $B(i)_j$ or $B(i)_{j+1}$. Therefore $d(B(i)_j, B(i+1)_k) < \varepsilon_i$.

Thus, there is a homeomorphism $g:M_1 \to M_2$. If we let g be the homeomorphism described in theorems 4.1 and 4.2, then $g(p_1) = p_2$ and $g(q_1) = q_2$. ■

The continua M_1 and M_2 in Theorem 4.3 were constructed by taking the intersection of crooked sequences of chains from p_1 to q_1 and from p_2 to q_2, respectively, and, hence, are pseudo arcs. Since the crooked sequences were arbitrarily chosen, we can state the following corollary.

COROLLARY 4.4: Any two pseudo arcs are homeomorphic.

We now have all that is needed to prove the pseudo arc is homogeneous. If we are given two points p_1 and p_2 in the pseudo arc and can construct two crooked sequences of chains from p_1 to q_1 and from p_2 to q_2, for some points q_1 and q_2, then we can apply Theorem 4.3. Suppose $\mathcal{D}_1, \mathcal{D}_2, \ldots$ is a crooked sequence of chains used to construct M, can some sequence of $\mathcal{D}_i$'s be consolidated into a crooked sequence of chains from p_1 to q_1? Theorem 3.7 might be useful, but it can only be applied to a chain $\mathcal{D}_j$ whose subchain from p_1 to q_1 intersects both endlinks of some $\mathcal{D}_k$, $(k < j)$. Let $\mathcal{D}_i(p_1, q_1)$ denote the subchain of $\mathcal{D}_i$ from the point p_1 to the point q_1. It is easy to see that not every subchain $\mathcal{D}_i(p_1, q_1)$ meets the requirement of Theorem 3.7. A subchain of $\mathcal{D}_2$ from p_1 to q_1, for example, might only be 2 links long and not meet either end link of $\mathcal{D}_1$. However, if q_1 is picked carefully, by which we mean q_1 is in a different composant of M than p_1, then given any integer k we will be able to find an integer j so that the subchain $\mathcal{D}_j(p_1, q_1)$ intersects $D(k)_1$ and $D(k)_L$, the endlinks of $\mathcal{D}_k$. However, before we can prove this, we need the following lemma.

Lemma 4.5: Let $\mathcal{D}_1, \mathcal{D}_2, ..$ be a sequence of 1/i - chain covers of a chainable continuum X and let p,q be points in X. Then the limit of any convergent subsequence of a sequence of sets $Cl(\cup \mathcal{D}_1(p, q))$, $Cl(\cup \mathcal{D}_2(p, q))$, . . . is a subcontinuum of X containing p and q.

Proof: In this proof we will be using the hyperspace $2^X = \{A \subset X \mid A \neq \emptyset$ and A is closed$\}$ with the Hausdorff metric. Specifically, we will need the fact that 2^X is compact when X is compact. Proof of this can be found on page 7 of Nadler[15].

The set $Cl(\cup \mathcal{D}_n(p, q))$ is closed in X for every integer n. Thus, $\langle Cl(\cup \mathcal{D}_n(p, q))\rangle$ is a sequence in 2^X. Since 2^X is compact, there is a convergent subsequence $\langle Cl(\cup \mathcal{D}_{n(k)}(p, q))\rangle$, say it converges to Y.

Claim: Y is a continuum containing p and q.

Since $Y \in 2^X$, Y is closed and compact in X and it is easy to see that $p, q \in Y$. Thus, to prove our claim we must show Y is connected. Suppose this is not true and $Y = H \cup K$, where H and K are none empty, disjoint, closed sets. Let $\varepsilon = \inf\{d(h, k) \mid h \in H$ and $k \in K\}$. Since $H \cap K = \emptyset$, $\varepsilon > 0$. Note that this ε is the shortest distance between the set H and the set K, which is much different than the Hausdorff distance between sets.

The fact that $\langle Cl(\cup \mathcal{D}_{n(k)}(p, q))\rangle$ converges to Y implies that there exists an integer h such that $d(Y, Cl(\cup \mathcal{D}_{n(i)}(p, q))) < \varepsilon/4$ for every $i > h$. Let $j > h$ such that $n(j) > 4/\varepsilon$. Then the diameter of each link of $\mathcal{D}_{n(j)} < \varepsilon/4$.

If $D \in \mathcal{D}_{n(j)}(p, q)$, then $D \subset N_{\varepsilon/4}(H)$ or $D \subset N_{\varepsilon/4}(K)$. But the shortest distance between $N_{\varepsilon/4}(H)$ and $N_{\varepsilon/4}(K)$ is greater than $\varepsilon/4$, which implies $\mathcal{D}_{n(i)}(p, q)$ is not a chain. This is a contradiction.

Therefore, Y is a continuum containing p and q. ■

Lemma 4.6: Let $\mathcal{D}_1, \mathcal{D}_2, \ldots$ be a crooked sequence of chains from x to y. Let $M = \bigcap_i(\mathrm{Cl}(\bigcup \mathcal{D}_i))$, be a pseudo arc. Let p and q be points in different composants of M. Then for every integer k there is an integer j such that the subchain $\mathcal{D}_j(p, q)$ meets both $D(k)_1$ and $D(k)_L$.

Proof: Let k be any integer. By Lemma 4.5 some subsequence of $\mathrm{Cl}(\bigcup \mathcal{D}_{k+1}(p, q))$, $\mathrm{Cl}(\bigcup \mathcal{D}_{k+2}(p, q)), \ldots$ converges to a subcontinuum of M containing p and q. But p and q are in different composants so this continuum must be M. But, $D(k)_1 \cap M \neq \emptyset$ and $D(k)_L \cap M \neq \emptyset$. Thus, for some $j > k$, $\bigcup \mathcal{D}_j(p, q) \cap D(k)_1 \neq \emptyset$ and $\bigcup \mathcal{D}_j(p, q) \cap D(k)_L \neq \emptyset$. ■

THEOREM 4.7: The pseudo arc is homogeneous.

Proof: In the first part of the proof we will to show that given any two points p and q in different composants of M, we can construct a crooked sequence of chains from p to q whose intersection is M.

Let $\mathcal{D}_1, \mathcal{D}_2, \ldots$ be a crooked sequence of chains from one point in the plane to another. Let $M = \bigcap_i(\mathrm{Cl}(\bigcup \mathcal{D}_i))$. Then M is a pseudo arc.

Let p and q be points in different composants of M. By Lemma 4.6 and Theorem 3.4 there is an integer $h > 2$ and a chain $\mathcal{E}_1$ from p to q, which is a limited consolidation of $\mathcal{D}_h$ with respect to $\mathcal{D}_2$, such that the diameter of each link of $\mathcal{E}_1$ is less than 1.

[Remark: Because theorem 3.4 uses limited consolidations, the size of the resulting links can be controlled. For example, if $h > 2$, then $\mathcal{D}_h \stackrel{c}{\leq} \mathcal{D}_2$. Furthermore, if $\mathcal{C}_1$ is a limited consolidation of $\mathcal{D}_h$ with respect to $\mathcal{D}_2$, then each link of $\mathcal{C}_1$ is contained in at most 2 links of $\mathcal{D}_2$, each of which has diameter less than 1/2. Hence, the diameter of each link of $\mathcal{C}_1$ is less than 1.]

Next we will construct a chain $\mathcal{C}_2$ that is crooked in $\mathcal{C}_1$ and has links whose diameters are less than 1/2. Let i be an integer such that $i > h$. Then the diameter of each link of $\mathcal{D}_i$ is less than 1/2. Which implies that there exists an integer j such that the closure of every pair of adjacent links of $\mathcal{D}_j$ is contained in a link of $\mathcal{D}_i$.

By Lemma 4.6 and Theorem 3.4, there is an integer $k > j$ and a chain $\mathcal{C}_2$ from p to q which is a limited consolidation of $\mathcal{D}_k$ with respect to $\mathcal{D}_j$. Each link of $\mathcal{C}_2$ is contained in at most 2 links of $\mathcal{D}_j$ which in turn are contained in only one link of $\mathcal{D}_i$. Hence, each link of $\mathcal{C}_2$ has diameter less than 1/2. Since $\mathcal{D}_i \leq \mathcal{D}_h$ and $\mathcal{D}_h \underset{c}{\leq} \mathcal{C}_1$, $\mathcal{D}_i \leq \mathcal{C}_1$ by Theorem 3.1. But $\mathcal{C}_2 < \mathcal{D}_i$ so $\mathcal{C}_2 \stackrel{c}{\leq} \mathcal{C}_1$ by Theorem 3.2.

This process can be continued to yield a crooked sequence of chains $\mathcal{C}_1, \mathcal{C}_2, \ldots$ from p to q. Since each $\mathcal{C}_i$ is a consolidation of some $\mathcal{D}_j$, each $\mathcal{C}_i$ covers M and therefore, $\bigcap_i(\mathrm{Cl}(\bigcup \mathcal{C}_i)) = M$.

Now suppose p_1 and p_2 are two points of M. Let q_1 and q_2 be points in composants not containing p_1 and p_2 respectively. Using the above result there is a crooked sequence of chains $\mathcal{T}_1, \mathcal{T}_2, \ldots$ from p_1 to q_1 and a crooked sequence of chains $\mathcal{D}_1, \mathcal{D}_2, \ldots$ from p_2 to q_2 such that $\bigcap(\mathrm{Cl}(\bigcup \mathcal{T}_i)) = M$ and $\bigcap(\mathrm{Cl}(\bigcup \mathcal{D}_i)) = M$. Therefore by Theorem 4.3 there is a homeomorphism $g{:}M \to M$ such that $g(p_1) = p_2$ (and $g(q_1) = q_2$). Hence, M is homogeneous. ■

The final two theorems are useful in the classification of plane continua as pseudo-arcs. In the first of these two theorems we will show that every hereditarily indecomposable chainable continuum is a pseudo arc. The key step is to show that for a given sequence of chain covers of an hereditarily indecomposable continuum there must be some subsequence that is a crooked sequence of chains. Once this is done we can easily show that this must be a pseudo arc.

THEOREM 4.8: Let M be a nondegenerate, compact, hereditarily indecomposable, chainable continuum. Then M is a pseudo arc.

Proof: Since M is chainable, there is a sequence of chains $\mathcal{C}_1, \mathcal{C}_2, \ldots$ such that the diameter of each link of $\mathcal{C}_i$ is less that 1/i and $\mathcal{C}_{i+1} < \mathcal{C}_i$.

Claim: There exists an integer t such that $\mathcal{C}_t \lesssim \mathcal{C}_1$.

Suppose t does not exist. Then there exists links $C(1)_h$ and $C(1)_k$, where $k - h > 2$, such that there are an infinite number of chains $\mathcal{C}_j$ that do not meet the requirement of being crooked between $C(1)_h$ and $C(1)_k$, i.e., there exists links $C(j)_u$ and $C(j)_v$ of $\mathcal{C}_j$ contained in $C(1)_h$ and $C(1)_k$, respectively, such that if $C(j)_r \subset C(1)_{k-1}$, for some r between u and v, then $C(j)_i \not\subset C(1)_{h+1}$ for every i between r and v.

[Remark: The number of possible chains $\mathcal{C}_j$ must be infinite for some h and k since there is a finite number of links and hence a finite number of combinations of h and k, but an infinite number of chains to choose from.]

Let $a_1, a_2, \ldots$ be the increasing sequence of integers such that, if $j = a_i$ for some i, $\mathcal{C}_j$ does not meet the requirement of being crooked between $C(1)_h$ and $C(1)_k$. For every $j = a_1, a_2, \ldots$, let u, v, and r be integers as defined in the above paragraph.

Without loss of generality, we assume $u < r < v$. Let $W_j = \cup\mathcal{C}_j(u, r)$ and $V_j = \cup\mathcal{C}_j(r, v)$.

By Lemma 4.4, we can find an increasing subsequence $b_1, b_2, \ldots$ of $a_1, a_2, \ldots$

such that $\langle W_{bi} \rangle$ converges to some continuum W. $\langle V_{bi} \rangle$ may not converge; however there is a subsequence $c_1, c_2, \ldots$ of $b_1, b_2, \ldots$ such that $\langle V_{ci} \rangle$converges to a continuum V. Since $\langle W_{bi} \rangle$ converges to W , so does $\langle W_{ci} \rangle$.

Since $W_{ci} \cap V_{ci} \neq \emptyset$ for every $i = 1, 2, \ldots$, $W \cap V \neq \emptyset$ and, therefore, $W \cup V$ is a continuum. By the way that r was picked $W \cap D(1)_k = \emptyset$ and $V \cap D(1)_k \neq \emptyset$, which implies $V \not\subset W$. Also, $W \cap D(1)_h \neq \emptyset$ and $V \cap D(1)_h = \emptyset$, which implies $W \not\subset V$. This means $W \cup V$ is a decomposable subcontinuum of M, which is a contradiction. Therefore there exists t such that $\mathcal{C}_t \lesssim \mathcal{C}_1$.

Repeating the argument we can find an integer s such that $\mathcal{C}_s \lesssim \mathcal{C}_t$. By continuing this process we can find a crooked sequence of chains $\mathcal{D}_1, \mathcal{D}_2, \ldots$ which is a subsequence of $\mathcal{C}_1, \mathcal{C}_2, \ldots$, such that $\bigcap(\mathrm{Cl}(\bigcup \mathcal{D}_i)) = M$. By the definition, M is a pseudo arc. ■

There are two facts that follow from this last theorem. First, it verifies the statement in the introduction that Knaster's hereditarily indecomposable continuum is indeed a pseudo-arc, and as he suspected, it is also homogeneous. Second, it is easy to see that every subcontinuum of a pseudo arc is hereditarily indecomposable and chainable. Thus, as a result of theorem 4.8, each subcontinuum of a pseudo arc is a pseudo arc and we can state the following corollary to theorem 4.8.

COROLLARY 4.9: The pseudo arc is hereditarily equivalent.

In the final theorem we will prove that every homogeneous, chainable continuum is a pseudo-arc. The proof will involve the claim that every point of such a continuum is an endpoint. Recall from the proof Theorem 4.7 that if p and q are points in different composants of a pseudo arc, then we can construct a crooked sequence of chains from p to q. In our definition of a crooked sequence of chains we stated that p and q are endpoints. Here we give the formal definition of endpoint. Notice that this definition does not contradict our previous statement, since for any $\varepsilon > 0$, there is an integer i such that $1/i < \varepsilon$. Hence, $\mathcal{D}_i$ is an ε-chain cover with p and q in its endlinks.

DEFINITION: A point p of a chainable continuum X is an **endpoint** if and only if for every $\varepsilon > 0$ there exists an ε-chain cover of X such that p is in an endlink.

LEMMA 4.10: Let X be a chainable continuum and let $p \in X$ be an endpoint. If H and K are subcontinua of X containing p, then $H \subset K$ or $K \subset H$.

Proof: Let p be an endpoint in a continuum X and let H and k be subcontinua, such that $p \in H \cap K$, $H \not\subset K$, and $K \not\subset H$. Then there exists points $a \in H - K$ and $b \in K - H$. Let $\varepsilon_1 = \inf\{d(a, k) \mid k \in K\}$ and $\varepsilon_2 = \inf\{d(b, h) \mid h \in H\}$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

Since p is an endpoint, there is an ε - chain cover $\mathcal{E}$ of X such that $p \in E_1$. Let n, m be the least integers such that $H \subset \bigcup\mathcal{E}(1, n)$ and $K \subset \bigcup\mathcal{E}(1, m)$. This implies that $E_n \cap H \neq \emptyset$ and $E_m \cap K \neq \emptyset$.

Either $n \leq m$ or $n \geq m$. Without loss of generality, suppose $n \leq m$. Let $E_i \in \mathcal{E}(1, n)$ such that $a \in E_i$. Since $\operatorname{diam}(E_i) < \varepsilon$, $E_i \cap K = \emptyset$. But, $E_1 \cap K \neq \emptyset$ and $E_m \cap K \neq \emptyset$, which implies K is not connected; a contradiction. Therefore either $K \subset H$ or $H \subset K$. ■

THEOREM 4.11: Every homogeneous, nondegenerate, chainable continuum is a pseudo arc.

Proof: Let N be a homogeneous, nondegenerate, chainable continuum. Using the following claim we will show that every point in N is an endpoint. We will then show that this means N is hereditarily indecomposable, which, by Theorem 4.7, means N is a pseudo arc.

Claim: Let $x \in N$ and let U be an open neighborhood containing x. Then for every $\varepsilon > 0$ there is an ε-chain cover of N one of whose endlinks is a subset of U.

Since N is chainable, for every integer i there exists an 1/i - chain cover $\mathcal{E}_i$ of N. Let $q_i \in N$ be a point in $E(i)_1$. Since N is compact, the sequence $q_1, q_2, \ldots$ has a subsequence that converges to a point $q \in N$.

Let U be an open set containing q, and let $\varepsilon > 0$ be given. Since there is some subsequence of $\langle q_i \rangle$ that converges to q and $\text{diam}(E(i)_1) \to 0$ as $i \to \infty$, there exists an integer $j > 1/\varepsilon$ such that $E(j)_1 \subset U$. In other words, there is an ε - chain cover of N one of whose endlinks is contained in U.

Because N is homogeneous, all points of N have this property and the claim is proved.

Using the claim, if $\mathcal{D}_i$ is a 1/i - chain cover of N and $p_i \in D(i)_1$, then there is a 1/(i+1) - chain cover $\mathcal{D}_{i+1}$ of N such that $D(i+1)_1 \subset D(i)_1$. Since $D(i+1)_1 \neq \emptyset$, let $p_{i+1} \in D(i+1)_1$ and we can find a 1/(i+2) chain cover $\mathcal{D}_{i+2}$ such that $D(i+2)_1 \subset D(i+1)_1$. Continuing this process we can find a sequence of 1/i - chain covers $\mathcal{D}_1, \mathcal{D}_2, \ldots$ such that $D(i+1)_1 \subset D(i)_1$. Let $p = \bigcap D(i)_1$. Then p is an endpoint. Since N is homogeneous, every point of N is an endpoint.

Finally, we want to show that N is hereditarily indecomposable. Let X be a subcontinuum of N. Suppose $X = H \cup K$ where H an K are subcontinua such that $H \not\subset K$ and $K \not\subset H$. Let $p \in H \cap K$. But, since p is an endpoint, $H \subset K$ or $K \subset H$ by Lemma 4.10. Hence, X is indecomposable. Therefore, N is hereditarily indecomosable and, by Theorem 4.8, N is a pseudo arc. ■

We have now reached the goal of this thesis. The construction of the pseudo-arc has been explained and we have shown that it is hereditarily indecomposable, homogeneous, and hereditarily equivalent. We also proved two theorems that provide a means to classify some plane continua. There are, however, many interesting facts regarding pseudo-arcs that we did not include. A few examples, which the reader may wish to investigate are listed below.

I. Bellamy[1] and Henderson [9]proved that a pseudo- arc can be constructed using inverse limits with the appropriate bonding maps. Bellamy[1] also demonstrated that any hereditarily indecomposable continuum could be mapped onto a pseudo-arc.

II. Bing and Jones[6] found another homogeneous plane continuum called a circle of pseudo-arcs. This brought the total number of known homogeneous plane continua to three. The question is, are there any others?

III. Every chainable continuum is the continuous image of a pseudo-arc.

IV. In a talk at the March 1984 Topology Conference, Wayne Lewis asked the following questions: 1) is every indecomposable, hereditarily equivalent continuum chainable?, and 2) is every homogeneous, hereditarily indecomposable continuum chainable? He mentioned that it is true in both cases if for every $\varepsilon \geq 0$ the continuum has an open ε-cover; i.e. the continuum is treelike.

APPENDIX The Pattern of a Maximal Crooked Chain

In this appendix we will discuss a specific pattern. This pattern is special because it follows a crooked path, and if one tries to consolidate it the resulting chain is no longer crooked. The following definition gives some properties of a chain following this pattern.

DEFINITION: A chain $\mathfrak{F}$ is a **maximal crooked chain** in a chain $\mathcal{E} = \{E_1, E_2, .. E_n\}$ if $\mathfrak{F} \precsim \mathcal{E}$, $F_1 \subset E_1$, $F_L \subset E_n$, $\mathfrak{F}(2, L - 1) \cap (E_1 \cup E_n) = \emptyset$, and no proper consolidation of $\mathfrak{F}$ has these properties.

One fact that is immediately apparent is that no two adjacent links of $\mathfrak{F}$ can be contained in the same link of $\mathcal{E}$. If this were to occur, then we could simply consolidate the two links together and still have a chain with all of the properties.

By finding the pattern that the maximal crooked chain $\mathfrak{F}$ follows in $\mathcal{E}$, we will answer two questions. First, for a given number of links in $\mathcal{E}$ how many links will $\mathfrak{F}$ have? Second, what is the function that defines the pattern? To answer the first question we will consider a simple example: the case where $\mathcal{E}$ has 5 links.

To construct $\mathfrak{F}$, a maximal crooked chain in $\mathcal{E}$, start with $F_1 \subset E_1$. As we continue the construction we must not place two adjacent links of $\mathfrak{F}$ in a single link of $\mathcal{E}$; hence, $F_2 \subset E_2$. We must also make $\mathfrak{F} \precsim \mathcal{E}$, so after placing F_3 in E_3 we cannot have $F_4 \subset E_4$, since this would mean $\mathfrak{F}(1,4)$ would not be crooked. Figure B-1 shows how the completed chain might look. Notice that any consolidation would change $\mathfrak{F}$ in such a way that either $\mathfrak{F}$ is not contained in $\mathcal{E}$ or $\mathfrak{F}$ is not crooked in $\mathcal{E}$. There are 13 links in $\mathfrak{F}$.

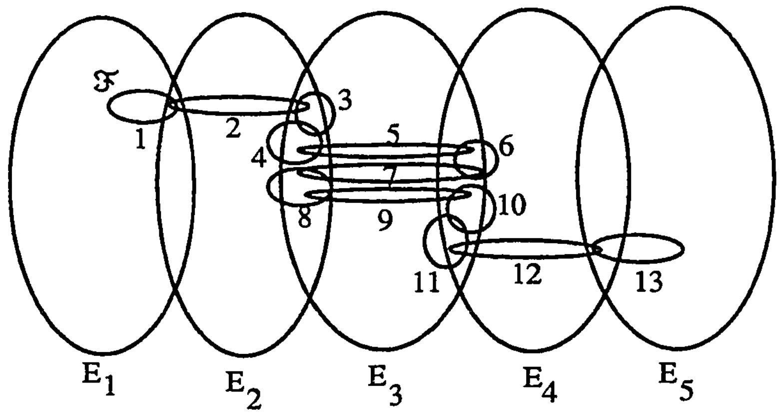


Figure B-1

How many links would $\mathcal{F}$ have if $\mathcal{E}$ had 6 links? The answer can be found by recalling from chapter 2 that the construction of crooked chains is a recursive process. If $\mathcal{E}$ has 6 links and $\mathcal{F} \lessapprox \mathcal{E}$, then $\mathcal{F} = \mathcal{F}(1, r) \cup \mathcal{F}(r, s) \cup \mathcal{F}(s, L)$ where $F_r \subset E_5$ and $F_s \subset E_2$, as shown in Figure B-2. The subchain $\mathcal{F}(1, r)$ is going to have 13 links since it is contained in 5 links of $\mathcal{E}$, as is the subchain $\mathcal{F}(s, L)$. The subchain $\mathcal{F}(r, s)$ is contained in 4 links of $\mathcal{E}$, so it will have 6 links. The total number of links in $\mathcal{F}$ is 13 + 13 + 6 -2 (we subtract 2 since F_r and F_s are each contained in 2 of the subchains, but should be counted only once). In general, if $\lambda(n)$ is the number of links in the maximal crooked chain $\mathcal{F}$ in $\mathcal{E}$ when $\mathcal{E}$ has n links, then $\lambda(n) = 2\lambda(n - 1) + \lambda(n - 2) - 2$.

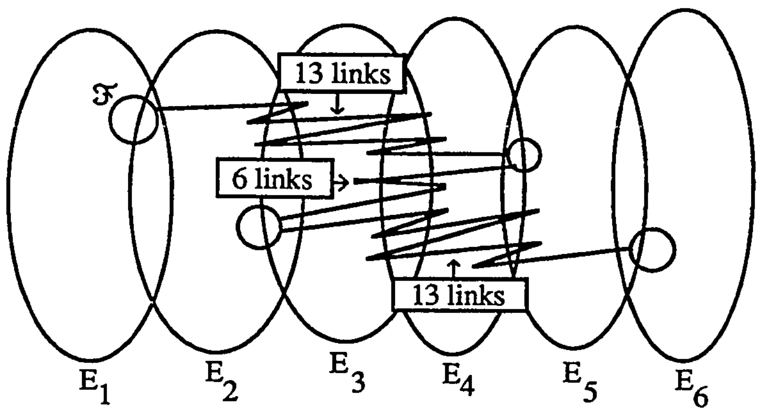


Figure B-2

Next we would like to find the pattern that $\mathfrak{F}$ follows in $\mathcal{E}$. The pattern of a maximal crooked chain $\mathfrak{F}$ in $\mathcal{E}$, denoted by f_c, is a function from $\{1, 2, \ldots, \lambda(n)\}$ onto $\{1, 2, \ldots, n\}$. First, observe that, no matter how many links $\mathcal{E}$ has, the pattern always starts the same. That is to say, if we increase the length of $\mathcal{E}$ from n to n+1 links, the pattern of the first $\lambda(n)$ links of $\mathfrak{F}$ will remain the same. Thus, suppose we want to find in which link of $\mathcal{E}$ some link F_i is contained; in other words, we want to find the value of $f_c(i)$. If we first find r such that $i \leq \lambda(r)$ then we only need to find the pattern in the first r links of $\mathcal{E}$. Actually we can say $\lambda(r - 1) < i \leq \lambda(r)$ since if $i \leq \lambda(r-1)$ we would only need to consider the first r-1 links of $\mathcal{E}$. For example, if i = 10 then r = 5 since $\lambda(4) = 6$ and $\lambda(5) = 13$. Thus, no matter how long $\mathcal{E}$ is all we need to consider is the first 5 links of $\mathcal{E}$ and we would find $F_{10} \subset E_4$ or $f_c(10) = 4$, as in Figure B-1.

Again, using the example in Figure B-1 where $\mathcal{E}$ has 5 links, notice that the subchains $\mathfrak{F}(1, 6)$ and $\mathfrak{F}(8, 13)$ have the same shape. The only difference is that $\mathfrak{F}(8, 13)$ starts in E_2. Also, the subchain $\mathfrak{F}(8,7)$ follows the same pattern as the subchain $\mathfrak{F}(8,9)$. To find the pattern now is just a matter of finding the correspondence between the links of $\mathfrak{F}(7, 13)$ and $\mathfrak{F}(1, 6)$. The link F_8 is the key to finding this correspondence, because the links of $\mathcal{F}(7, 13)$ are related to F_8 in the same way that the links of $\mathcal{F}(1, 6)$ are related to F_1. What must be done first, then, given some arbitrary link F_i of $\mathcal{F}(7, 13)$, is to find the distance in links from F_8 to F_i; this is $|8 - i|$. By adding 1 to this number we get the exact link in $\mathcal{F}(1, 6)$ to which F_i corresponds. For example, if $i = 10$, F_{10} corresponds to F_3, because $1 + |8 - 10| = 3$. Hence, by the way in which the chains $\mathcal{F}(1, 6)$ and $\mathcal{F}(8, 13)$ are related, $f(10) = 1 + f(3) = 4$ and, therefore, $F_{10} \subset F_4$.

In general, once we have found r, the key link is the $(\lambda(r - 2) + \lambda(r - 1) - 1)^{th}$ link and $f_c(i) = 1 + f_c(1 + |(\lambda(r - 2) + \lambda(r - 1) - 1) - i|)$. Since $\lambda(n)$ is the minimum number of links needed to construct a chain of maximal crooked consolidation, the pattern f_c is the only pattern such a chain could follow. Any deviation from this pattern would require more links or result in a chain that does not meet the requirements.

There are two reasons why the maximal crooked chain $\mathcal{F}$ in $\mathcal{C}$ is of interest. First, the maximal crooked chain tells us something about how difficult it would be to draw more than 2 or 3 chains in a crooked sequence of chains. This is because the number of links required gets very large, very fast. For example, let $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots$ be a crooked sequence of chains and let $\mathcal{D}_1$ have 4 links. Since $\mathcal{D}_2 \lesssim \mathcal{D}_1$ and $\lambda(4) = 6$, $\mathcal{D}_2$ must have at least 6 links. Furthermore, $\mathcal{D}_3 \lesssim \mathcal{D}_2$ and $\lambda(6) = 30$, therefore, $\mathcal{D}_3$ has at least 30 links. Without the aid of a computer, try to fill in the blanks in the next step. Since $\mathcal{D}_4 \lesssim \mathcal{D}_3$ and $\lambda(30) =$ ____, $\mathcal{D}_4$ has ____ links. If you are able to figure this out, try and find the number of links in $\mathcal{D}_5$. Remember, because $\lambda(n)$ gives us the fewest number of links needed to construct a crooked chain, these numbers in the last example are minimums!

The second reason this pattern is interesting is given by the following theorem. Contrast this theorem with Theorem 3.7.

THEOREM B.1: If $\mathcal{D} \lesssim \mathcal{E} = \{E_1, E_2, \ldots, E_n\}$, and for some i and j, $D_i \subset E_1$ and $D_j \subset E_n$, then there is a chain $\mathcal{F} \gtrsim_c \mathcal{D}$ which is of maximal crooked chain in $\mathcal{E}$.

Proof: (by induction on n, the length of $\mathcal{E}$)

It is easy to see that the theorem holds for $n = 3$ and $n = 4$. Suppose the theorem holds for $n \leq m-1$, for some integer $m \geq 4$. We will show that the theorem holds for $n = m$.

Since only the endlinks of $\mathcal{F}$ are to be contained in the endlinks of $\mathcal{E}$, let $F_1 = \bigcup\{D_i \mid D_i \subset E_1\}$ and $F_{\lambda(n)} = \bigcup\{D_i \mid D_i \subset E_n\}$. This implies $\{D_i \mid D_i \subset E_2$ and $D_i \cap E_1 \neq \emptyset\} \subset F_2$ and $\{D_i \mid D_i \subset E_{n-1}$ and $D_i \cap E_n \neq \emptyset\} \subset F_{\lambda(n)-1}$.

All of the remaining links of $\mathcal{D}$ are contained in subchains that intersect one or both of the endlinks of $\mathcal{E}$, but whose interior links do not meet E_1 or E_n. We will show how the links of one such chain are placed into the links of $\mathcal{F}$.

Let $\mathcal{D}(r, s)$ be a subchain of $\mathcal{D}$ such that $D_r \cap E_1 \neq \emptyset$, $D_s \cap E_n \neq \emptyset$ and no interior link of $\mathcal{D}(r, s)$ meets E_1 or E_n. Then there exist links $D_j \subset E_{n-1}$ and $D_k \subset E_2$ such that $\mathcal{D}(r, s) = \mathcal{D}(r, j) \cup \mathcal{D}(j, k) \cup \mathcal{D}(k, s)$.

Since no interior links of $\mathcal{D}(r, s)$ intersect E_1 or E_n, $\mathcal{D}(r, j) < \mathcal{E}(1, n-1)$, $\mathcal{D}(j, k) < \mathcal{E}(2, n-1)$ and $\mathcal{D}(k, s) < \mathcal{E}(2, n)$. Each of these subchains is contained in m - 1 or fewer links, so we can perform the three following consolidations:

1. By the inductive hypothesis, there exists a maximal crooked chain $\mathcal{F}_1$ in $\mathcal{E}(1, m-1)$ which is a consolidation of $\mathcal{D}(r, j) \cup \{$some link $D \subset E_1\}$.

2. By the inductive hypothesis, there exists a maximal crooked chain $\mathcal{F}_2$ in $\mathcal{E}(2, m-1)$ which is a consolidation of $\mathcal{D}(j, k)$.

3. By the inductive hypothesis, there exists a maximal crooked chain $\mathcal{F}_3$ in $\mathcal{E}(2, m)$ which is a consolidation of $\mathcal{D}(k, s) \cup \{$some link $D \subset E_m\}$.

It is necessary to add links of $\mathcal{D}$ contained in E_1 and E_m so that $F(1)_1 \subset E_1$ and $F(3)_{\lambda(m-1)} \subset E_m$. If we join these three chains together by letting $\mathfrak{F}_1$ be the first links, $\mathfrak{F}_2$ be the middle links, and $\mathfrak{F}_3$ be the last links, then the resulting chain will follow the desired pattern in $\mathcal{E}$.

As mentioned previously, all of the other links of $\mathcal{D}$ are contained in subchains which, like $\mathcal{D}(r, s)$, meet both E_1 and E_m, or they are contained in subchains that meet only one of the end links of $\mathcal{E}$. If the subchain is like $\mathcal{D}(r, s)$, then it can be consolidated in the same way. On the other hand, if the subchain only meets one of E_1 or E_m, then it can be consolidated to follow the same pattern as $\mathfrak{F}_1$ or $\mathfrak{F}_3$, respectively. ∎

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